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On the Effect of Stochastic Fluctuations in the Dynamics of the Lifshitz–Slyozov–Wagner Model

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In this paper the dynamics of a system of spherical particles that fill a small volume fraction of the space and that evolves in a concentration field is discussed. Corrections to the Lifshitz–Slyozov–Wagner (LSW) model that take into account the stochastic character of the problem are computed. It is proved, under suitable smallness assumptions for the volume fraction filled by the particles, that the effect of these corrections does not modify much the dynamics of the self-similar solutions of the LSW system of equations.

KEY WORDS: Ostwald ripening; self-similar solutions; stochastic fluctuations; intermediate asymptotics.

1. INTRODUCTION

The classical Lifshitz–Slyozov–Wagner model (LSW from now on) describes the distribution of radii of a set of particles whose size changes in time due to diffusion (cf. refs. 5 and 16). In the LSW model it is assumed that the particles remain spherical during their evolution and also that the diffusion field is quasistatic. General reviews on the LSW theory are refs. 14 and 15.

The LSW model admits a family of self-similar solutions that are supposed to describe the long time behaviour of the LSW system of equations (cf. ref. 5). For each given density of particles the family of self-similar solutions depends on one parameter. A natural problem is to select among this family the solution (if any) that would describe the distribution of radii during the coarsening stage of the aggregate of particles. This selection cannot be made just in the framework of the pure LSW theory, due to some recent results about the stability of all the self-similar solutions that have

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been recently obtained in ref. 8 and that I will discuss in detail later. Let us remark for the moment that in the original article (cf. ref. 5) it was suggested that one particular self-similar solution of the LSW model should be the physically relevant one due to the stabilizing effect of collisions or "encounters" between particles. As a matter of fact, the LSW model considers the particles as isolated, and only introduces interaction between particles by means of an "average" field. The effect of "encounters" between particles is modelled by means of a collision term of kinetic type that is derived using some "ad hoc" assumptions about the dynamics of collisions between pairs of particles.

It is important to remark however that, although in principle collisions between particles could be determinant in selecting the "correct" solution among the family of self-similar solutions of the LSW model, there exist other possible regularizing effects that have not been taken into account in the paper.[5] Let us denote as ε_N the volume fraction of a container occupied by N particles. This quantity remains constant during the whole evolution of the system. In order to be able to consider the particles as isolated, as it is made in the derivation of the LSW model, the condition $\varepsilon_N \ll 1$ is required. However, other considerations restrict the validity of the LSW to smaller volume fractions. Indeed, the numerical simulations in ref. 6 show the existence of a transition in the long time behaviour of particle aggregates for volume fractions of order $\varepsilon_N \sim 1/N^2$. The existence of a natural transition if $\varepsilon_N \sim 1/N^2$ has also been observed in a more analytic way in ref. 9 with the use of homogeneisation arguments. The reason for the onset of this distinguished limit is the fact that in the quasistatic approximation the concentration field due to an isolated particle decays far away at the coulombian rate $1/r^2$. For volume fractions ε_N smaller than $1/N^2$ the local effects due to each particle can then be neglected. However, if ε_N of order $1/N^2$ or larger, the combined interaction of the coulombian-like concentrations described above becomes relevant. Although it seems that the LSW model can still be derived as a leading approximation in this case, some effects due to the stochastic fluctuations of the particles could be quite relevant as will be checked in Section 2. From the mathematical point of view the limit $\varepsilon_N \sim 1/N^2$ arises for reasons analogous to the so-called Debye–Hückel screening in the theory of electrolites, as has been pointed out in ref. 6.

It is the goal of this paper to understand the validity of the LSW theory for small volume concentrations of particles. In order to avoid the complications due to the Debye–Hückel screening effects discussed above, it is natural to consider in a first approximation volume fractions satisfying $\varepsilon_N \ll 1/N^2$. On the other hand, as it will be argued later, for volume fractions that are smaller than $1/N$ the probability of having encounters between particles is negligible if the particles are uniformly distributed. This clearly indicates that for the volume fractions considered in this paper ($\varepsilon_N \ll 1/N^2$), another regularising mechanism (if any) should be sought-for.

Before continuing the discussion about possible regularising effects for the LSW it is worth recalling some results that have been recently obtained in the articles.[7, 8] In these papers, the authors have established the mathematical well-posedness of the LSW model and they have analysed the long time asymptotics for the solutions of this problem. In particular has been obtained in refs. 7 and 8 that the long time asymptotics for the solutions of the LSW system depends very strongly on the asymptotic distribution of radii near the maximum radius (this analysis has been restricted to compactly supported distributions of radii). In particular, if the initial distribution of radii behaves as a power law, analogous to the distribution of a self-similar solution, the long time asymptotics of the corresponding solution of the LSW theory turns out to be such a self-similar solution. Non selfsimilar type of behaviours are also obtained.

As was stated before, these results show that finding a selection procedure for the physically relevant solution of the LSW theory using only the pure LSW system of equations and stability considerations is hopeless. In this paper, two different possible effects that could regularise the LSW model will be considered. First, we notice that the distribution density of particles is not a continuous one for a physical set of particles. Indeed, if the number of particles is finite, although large, the distribution function is really a stair-like function. We can assume that an initial distribution of particles with a given probability is assigned. In the long run, and with probability one, only one particle survives as it follows from the fact that particles with radius below the average disappear and particles with radius larger than the average growth (see ref. 8 for a more analytical description of this process). However, for ranges of times for which the number of particles is small, it does not make sense to use a continuum theory as LSW in order to describe the distribution of particles. A more natural question would be to establish if there is some intermediate range of times for which a large number of particles still exists, and which particular dynamics, among the many that are possible in the LSW model, takes place. We can establish this problem in a more precise form. If we take an initial distribution of probability, and assign the radii of N particles independently using such a distribution function, we obtain a distribution of radii for the finite set of particles. This last converges to the original one due to the classical law of the large numbers. Nevertheless, some brownian-like fluctuations appear in the distribution of radii for the finite set of particles. Since the long time asymptotics of the LSW model depend very strongly on the distribution of radii near the maximum radii, the possibility of an intermediate asymptotics for the discrete model that could be described by the classical LSW model cannot be ruled out immediately.

On the other hand, other possible regularisation mechanism is the fact that the mean field of concentration that is used in the derivation of the LSW theory is not the exact field that drives the dynamics of the particles. As a matter of fact, the mean field has to be corrected with an additional term that describes the fluctuations in the positions of the particles. Fluctuation terms of this type are relevant in some particular physical applications, for instance in stellar dynamics, where the corresponding fields have also a decay rate $1/r^2$ (cf. the extensive review[3]). In this particular case, these fluctuations due to the position would have the effect of introducing some degree of indetermination in the rate of change of the radius for particles with the same radii, something that usually has a stabilising effect in statistical mechanics problems.

The main goal of this paper is to check if the two effects mentioned above (fluctuations in the distribution of radii, and stochastic fluctuations in the rates of change for the radii due to the stochasticity on the positions of the particles) could have a role in selecting a particular solution of the LSW theory. The answer will be a negative one. More precisely, let us restrict our analysis to a range of times for which the remaining number of particles is still large enough as to allow a continuous description of the distribution of particles as it is made in the LSW theory. Assume also that the distribution of radii is chosen uniform in space, and according to any of the selfsimilar solutions of the LSW theory. Then, if the volume fraction ε_N satisfies $\varepsilon_N \ll 1/N^2$, neither of the stochastic terms described above is strong enough to modify in a sensible way the selfsimilar asymptotic behaviour.

The result mentioned above is in a clear contrast with the situation that takes place if we introduce in the model the effects due to the fluctuations on the size of the particles due to kinetic effects. In the articles,[11, 13] the LSW model has been considered as a limit case of the Becker–Döring system of equations. This last system is an infinite set of equations that describes the concentration of clusters of a given size. Such clusters can change their size by means of the addition or substraction of monomers. Formally, for large clusters, the Becker–Döring system approaches to the LSW model. However, the Becker–Döring model takes into account the existence of fluctuations of large clusters, and this effect regularises the LSW theory, as has been shown in ref. 13, and thus provides a way to pick the unique solution of the LSW model that has been selected in the original paper.[5] In some sense, the selection mechanism has some analogies with the selection of the solutions on hyperbolic equations by means of the zero viscosity limit, but the technicalities are very different in both cases.

The results described above seem to indicate that, at least for not very large volume fractions of particles, fluctuations are essential in making the selection of the correct solution of the LSW theory, in those situations where this distribution is actually observed.

The plan of this paper is the following. In Section 2 a correction of the LSW model that includes (to the leading order) the effect of the fluctuations on the positions of the particles is derived. In that Section will be also estimated the order of magnitude of the collisions between particles. This turns out to be small for the considered volume fractions. It will be also seen in this Section that drifting and deformations of the particles are negligible to the leading order. In Section 3 a number of statistical properties of the initial distribution of particles will be discussed. Section 4 contains some probabilistic estimates on the corrective terms due to that fluctuations. In Section 5 there is a proof of the fact that neither of the stochastic effects described significantly modifies a selfsimilar behaviour as far as the number of remaining particles is larger than $\log(N)$. For longer times, the effect of stochastic fluctuations should become important and a description of the distribution of particles by a continuous density should become meaningless. I have not attempted to describe this range of times in detail. Notice, however that the remaining set of particles during this stage is much smaller than the original one. There are several analogies between some of the main ideas in Section 5 as those in ref. 8, although from a technical point of view both approaches are rather different. Finally in Section 6 some conclusions and general discussion on the obtained results is provided.

2. COMPUTING THE FLUCTUATIONS OF THE RATE OF GROWTH

2.1. The LSW Model with Fluctuations

In this section we describe in detail the problem that we analyse in the paper and we make some preliminary approximations that allow us to write it in a more convenient way.

In the derivation of the LSW model it is assumed that a set of N particles evolves due to a diffusion field. Let us suppose that the particles are included in a cubic domain $\Omega = (0, L)^3$. In the usual LSW model it is also supposed that the diffusion field $c(x, t)$ is quasi stationary, and then solves the Laplace equation. Let us denote as $D_i(t)$ $i = 1,..., N$ the different particles. On the surface of each particle the Gibbs–Thomson law holds. Under these assumptions the growth of the particles is described by the following problem:

$$\Delta c = 0 \qquad \text{in } \Omega \Big\backslash \left(\bigcup_{i=1}^{N} D_i(t) \right) \tag{2.1}$$

$$c = -\sigma H \qquad \text{in } \bigcup_{i=1}^{N} \partial D_i(t) \tag{2.2}$$

$$V_n = -D \frac{\partial c}{\partial n} \qquad \text{in } \bigcup_{i=1}^{N} \partial D_i(t) \tag{2.3}$$

The coefficients $\sigma > 0$ and $D > 0$ in (2.2) and (2.3) are respectively the surface tension and the diffusion coefficient. We denote by H in (2.2) the mean curvature of each domain $\partial D_i(t)$. Finally V_n in (2.3) stands by normal velocity of growth. Problem (2.1)–(2.3) has to be solved for some suitable initial distribution of initial domains, with periodic boundary conditions at the boundaries of the region Ω, in order to avoid the influence of the walls in the dynamics of the problem.

From now on we will denote as ε_N the fraction of volume occupied by the particles. As stated in the introduction, we will assume in the whole paper that

$$\varepsilon_N N^2 \ll 1 \tag{2.4}$$

By convenience we adimensionalise the problem as follows:

$$\Phi = \frac{r_N}{\sigma} c \tag{2.5}$$

$$t' = \frac{D\sigma}{(r_N)^3} t \tag{2.6}$$

$$r_N = \left(\frac{|\Omega| \, \varepsilon_N}{N} \right)^{1/3} \tag{2.7}$$

In this new set of variables, the model (2.1)–(2.3) becomes:

$$\Delta \Phi = 0 \qquad \text{in } \Omega \Big\backslash \left(\bigcup_{i=1}^{N} D_i(t) \right) \tag{2.8}$$

$$\Phi = -r_N H \qquad \text{in } \bigcup_{i=1}^{N} \partial D_i(t) \tag{2.9}$$

$$V_n = -(r_N)^2 \frac{\partial \Phi}{\partial n} \qquad \text{in } \bigcup_{i=1}^{N} \partial D_i(t) \tag{2.10}$$

The main advantage of writing (2.1)–(2.3) in the form (2.8)–(2.10) is that Φ becomes of order one and, on the other hand, if we measure the sizes of the domains $D_i(t)$ using the natural length scale r_N, the variations of the domains is of the order of themselves if the time t' varies by quantities of order one. By notational simplicity, we will drop from now one the tilda from t'.

We will assume that the particles are spherical at the initial time. It will be checked in the derivation of the model that, under the assumption $\varepsilon_N N^2 \ll 1$, the particles remain spherical to the leading order during their evolution for long enough times, and we will make this hypothesis in the following. We shall denote the radius of the domain $D_i(t)$ as R_i, and in adimensional units we will write $\xi_i = R_i / r_N$.

If we assume that the particles are uniformly distributed the average distance between them would be of order $(1/N)^{1/3}$. Since, by assumption $\varepsilon_N \ll 1$, it then follows that $r_N \ll (1/N)^{1/3}$. It then turns out that most of the particles are quite separated from the others. We can make more precise this statement as follows. Due to the conservation of volume of the particle, the maximum volume that can have a particle during its evolution is ε_N. Let us take a spherical volume ε_N around each particle. The probability of having an empty intersection between these volumes is equal to:

$$P_0 = \prod_{\ell=1}^{N-1} \left(1 - \frac{\ell \varepsilon_N}{|\Omega|}\right) \tag{2.11}$$

where we have assumed that the particles are distributed independently and uniformly in the domain Ω. Since $\varepsilon_N N^2 \ll 1$ it then easily follows that P_0 can be approximated as:

$$P_0 \approx \exp\left(-\frac{\varepsilon_N N^2}{2\,|\Omega|}\right) \tag{2.12}$$

Using the fact that $\varepsilon_N N^2 \ll 1$ we deduce that the probability of intersection between the different domains is of order:

$$1 - P_0 \approx \frac{\varepsilon_N N^2}{2\,|\Omega|} \ll 1 \tag{2.13}$$

We can then assume that the different particles do not intersect during their evolution, except perhaps for some initial configurations with small probability in the limit of large numbers of particles. Actually, (2.13) is a rather rough estimate for the probability of intersection between domains during their evolution, since we have assumed that all the domains fill the maximum possible volume from the very beginning, and the vanishing of particles in time has not been taken into account. On the other hand, notice that if instead of using the total volume of the particles ε_N we take into account the volume for particle ε_N/N, we deduce that the probability of having intersections in the initial distribution of particles is small if $\varepsilon_N \ll 1/N$, as was indicated in the Introduction. In any case, we can safely assume that under the assumption (2.4), and supposing a uniform distribution of the particles, no collisions take place. This excludes the type of regularizing mechanism suggested in ref. 5 for these volume fractions (at least if initially, the particles are randomly distributed).

Since the domains $D_i(t)$ are spherical and, as indicated above, they are very far away from the others, we can approximate the field Φ near the ith particle as:

$$\Phi \approx \frac{A_i}{|x-x_i|} - B \tag{2.14}$$

for some suitable constants A_i, B. The constant B represents some kind of average field, and the term $A_i/|x-x_i|$ provides some boundary layer variation of the field due to the presence of the particle. The constants A_i can be determined by means of the boundary condition (2.9). It then follows that:

$$\Phi \approx -\frac{r_N(1-B\xi_i)}{|x-x_i|} - B \tag{2.15}$$

where we have used the fact that $H=1/R_i$.

Taking into account the boundary condition (2.10) as well as the approximation (2.15), we derive the following approximation for the rate of change of the radius of each particle:

$$\frac{d\xi_i}{dt} = -\frac{(1-B\xi_i)}{(\xi_i)^2} \tag{2.16}$$

In order to determine the time dependence of B, we use the fact that the system (2.8)–(2.10) preserves the total volume of the particles $\bigcup_{i=1}^N |D_i(t)|$, as can be easily checked by taking the derivative of this quantity, and then using (2.8), (2.10). In the particular case of spherical domains, it then follows that:

$$\sum_{i=1}^{N} (\xi_i)^3 = \mu_0 \tag{2.17}$$

where μ_0 is a fixed constant. Combining (2.16) and (2.17) we readily obtain that:

$$B(t) = \frac{\sum_{i=1}^{N} \chi(\xi_i)}{\sum_{i=1}^{N} \xi_i} \tag{2.18}$$

where from now on we denote $\chi(s) = 1$ if $s > 0$ and $\chi(0) = 0$.

The model (2.16), (2.17) is the classical LSW model. In the limit of a very large number of particles, a continuity equation in the space of radii for the density of particles with a given radius $f(\xi, t)$ can be derived. More precisely, if the fraction of remaining particles with a radius in the interval $(\xi, \xi + d\xi)$ is written as $dN(t)/N = f(\xi, t)\, d\xi$, the following continuity equation can be easily computed taking into account (2.16):

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \xi}\left(\left(-\frac{1}{\xi^2} + \frac{B(t)}{\xi}\right) f\right) = 0 \tag{2.19}$$

where in this continuous limit (2.17) becomes:

$$\int_0^\infty f(\xi, t)(\xi)^3\, d\xi = \frac{\mu_0}{N} \tag{2.20}$$

Recently, model (2.19) and (2.20) has been extensively studied (cf. refs. 7 and 8), It has been shown in these references that, for compactly supported densities, the long time dynamics of this model depends in a very sensitive way on the asymptotics of the initial data $f(\xi, 0)$ near the maximum radius. The goal of this paper is to derive a model more regular that the LSW one (2.16), (2.17), that takes into account the fluctuations of the velocities of the particles. The main consequence of this analysis will be to introduce some additional, stochastic terms in (2.16).

To be more precise, let us argue as follows. Suppose that the initial distribution of radii and positions for the particles is given according to the probability distribution:

$$dv(\xi, x) = \prod_{\ell=1}^{N} \left[\frac{f(\xi_\ell, 0)\, d\xi_\ell\, dx_\ell}{|\Omega|}\right] \tag{2.21}$$

where by assumption $f(\xi, 0)$ is a nonnegative probability measure.

Then the domains begin to evolve according to (2.8)–(2.10). Notice that in general the domains lose the radial shape for $t > 0$. However, it will be checked later that, except for a set of distributions with small probability the domains remain almost spherical to the lowest order.

As a next step, we compute the leading order correction due to the fluctuations on the positions of the particles. The driving force that produces the change in the radii of the particles can be decomposed in an average field plus a smaller, but rapidly fluctuating field. This last field varies very quickly in time, due to the vanishing of particles that are distributed in space in a stochastic manner. Let us write the evolution of the ith radius as:

$$\frac{d\xi_i}{dt} + \left(\frac{1}{(\xi_i)^2} - \frac{B(t)}{(\xi_i)} \right) = \sigma_i(t) \tag{2.22}$$

Our main interest in the next lines is to compute in detail the term $\sigma_i(t)$. To this end, we argue as follows. Formula (2.15) suggests the following global approximation for Φ:

$$\Phi \approx -B + Br_N \sum_{i=1}^{N} \frac{\xi_i}{|x - x_i|} - r_N \sum_{i=1}^{N} \frac{\chi(\xi_i)}{|x - x_i|} \tag{2.23}$$

We remark, however, that the right hand side of (2.23) does not satisfies the periodic boundary conditions that we are assuming in our problem. In order to avoid this difficulty, we introduce a Green's function $K(x, y)$ that is the unique function satisfying the following problem:

$$-\Delta_x K(x, y) = 4\pi\delta(x - y) - \frac{4\pi}{|\Omega|}, \qquad \text{for} \quad x \neq y \tag{2.24}$$

$$K(x, y) \sim \frac{1}{|x - y|} + O(|x - y|^2), \qquad \text{as} \quad x \to y \tag{2.25}$$

and with periodic boundary conditions at the boundary of the domain Ω. It is not hard to see that there exists a function $K(x, y)$ satisfying (2.24) and (2.25). Indeed, due to the periodic boundary conditions we need to impose a compatibility condition on the right hand side of (2.24), namely, the orthogonality to the constants, that explains the presence of the term $4\pi/|\Omega|$. On the other hand, solutions of (2.24) are not unique, since an arbitrary constant can be added. Such a constant is prescribed by means of the asymptotics (2.25) where not constant terms appear as $x \to y$. Noticed that the first corrective term in (2.25) is not linear but quadratic due to the symmetry of the problem satisfied by $K(x, y)$.

Instead of using (2.23), we shall approximate Φ as:

$$\Phi \approx -B + Br_N \sum_{i=1}^{N} \xi_i K(x, x_i) - r_N \sum_{i=1}^{N} \chi(\xi_i)\, K(x, x_i) \equiv W \tag{2.26}$$

Formula (2.26) provides a good approximation of Φ as far as the effects of the sums there remain small far away from the particles. At a first glance is not obvious why the second term of (2.26) should be a good approximation of the function Φ that is harmonic, since several of the terms in the definition of W_- are not harmonic (cf (2.24)). However, if we use the approximation (2.18) it immediately follows that Φ is solves Laplace's equation away from the points x_i, $i=1,\ldots,N$.

It is interesting to observe, that under the hypothesis $r_N N = (\varepsilon_N N^2)^{1/3} \ll 1$ the main contribution to the field near the particle x_ℓ is given by $K(x, x_\ell)$. Indeed, we can estimate, for instance, the term $r_N \sum_{i=1\, i\neq \ell}^{N} K(x, x_i)$ that can be approximated to the leading order as $r_N N\, |\Omega| \int K(x, y)\, dy$. The contribution of this term is small compared with B and $r_N K(x, x_\ell)$ if $r_N N = (\varepsilon_N N^2)^{1/3} \ll 1$. If this assumption falls, we would need to introduce additional terms due to the well known Debye–Hückel screening. We will not consider that range of parameters in this paper.

Taking into account (2.26), it is natural to introduce a corrective term V by means of the formula:

$$\Phi = W + N r_N V \tag{2.27}$$

Using (2.8), (2.9) as well as (2.26) it readily follows that V solves the following problem:

$$\Delta V = 0 \qquad \text{in } \Omega \Big\backslash \Bigg(\bigcup_{i=1,\, \chi(\xi_i)\neq 0}^{N} D_i(t) \Bigg) \tag{2.28}$$

$$\begin{aligned} V = {} & B\left(\frac{1}{r_N} - \frac{\xi_i}{|x-x_i|}\right) + \frac{1}{N}\left(\frac{1}{|x-x_i|} - H\right) \\ & + B\left(\frac{\xi_i}{|x-x_i|} - \xi_i K(x, x_i)\right) + \frac{1}{N}\left(K(x, x_i) - \frac{1}{|x-x_i|}\right) \\ & + \frac{1}{N}\left(\sum_{\ell=1,\, \ell\neq i}^{N} \chi(\xi_\ell)\, K(x, x_\ell) - B \sum_{\ell=1,\, \ell\neq i}^{N} \xi_\ell K(x, x_\ell)\right) \qquad \text{at } \partial D_i(t) \end{aligned} \tag{2.29}$$

where the boundary condition (2.29) is imposed only at those values of i where $\chi(\xi_i) \neq 0$. The first two terms at the right hand side of (2.29) are identically zero for radial particles. We will check later that the leading correction to (2.16) is radial, that means that those terms can be neglected to the first order. On the other hand, the third and fourth order terms on the right of (2.29) can be easily estimated using (2.25). This correction turns out to be of order $O((r_N)^2)$, which is very small compared with the last two terms of (2.29) that give contributions of order one. Summarizing, we can then replace (2.29) by:

$$V = \frac{1}{N}\left(\sum_{\ell=1,\,\ell\neq i}^{N} \chi(\xi_\ell)\, K(x, x_\ell) - B \sum_{\ell=1,\,\ell\neq i}^{N} \xi_\ell K(x, x_\ell) \right) \qquad \text{at } \partial D_i(t) \tag{2.30}$$

Taking into account that for most of the distributions of particles $|x_i - x_\ell| \gg r_N$, we can rewrite (2.30) to the leading order as:

$$V = \frac{1}{N} \sum_{\ell=1,\,\ell\neq i}^{N} (\chi(\xi_\ell) - B\xi_\ell)\, K(x_i, x_\ell) \equiv k_i \qquad \text{at } \partial D_i(t) \tag{2.31}$$

Equation (2.31) shows that, to the lowest order, function V takes a constant value on each particle.

Using (2.10) we obtain the following expression for the normal velocity at the surface of each particle:

$$\begin{aligned} V_n &= -(r_N)^2 \frac{\partial W}{\partial n} - (r_N)^3 N \frac{\partial V}{\partial n} \\ &= -(r_N)^3 \sum_{\ell=1}^{N} (B\xi_\ell - \chi(\xi_\ell))\, \nabla_x K(x, x_\ell) \cdot n - (r_N)^3 N \frac{\partial V}{\partial n}, \qquad \text{at } \partial D_i(t) \end{aligned} \tag{2.32}$$

To proceed further, we split the contribution due to $\partial W/\partial n$ on (2.32) in a part due to the ith particle and a remainder:

$$\begin{aligned} -(r_N)^2 \frac{\partial W}{\partial n} &= (r_N)^3 (B\xi_i - \chi(\xi_i)) \frac{(x - x_i)}{|x - x_i|^3} \cdot n \\ &\quad + (r_N)^3 (B\xi_i - \chi(\xi_i)) \left(\left(\nabla_x K(x, x_i) - \frac{(x - x_i)}{|x - x_i|^3} \right) \cdot n \right) \\ &\quad \times (r_N)^3 \sum_{\ell=1,\,\ell\neq i}^{N} (B\xi_\ell - \chi(\xi_\ell))\, \nabla_x K(x, x_\ell) \cdot n \end{aligned} \tag{2.33}$$

The first term on the right hand side of (2.33) is the leading term of the concentration field and the only one that contributes in the classical LSW theory. This term is radial on radial particles, and its nonradial contribution is proportional to the lack of radiality of the domains $D_i(t)$. This means that this term cannot be responsible for the deviation of radial behaviour of the particles. On the other hand, the last term on the right hand side of (2.33) can be approximated as:

$$(r_N)^3 \int_{\partial D_i(t)} \left[\sum_{\ell=1,\ell\neq i}^{N} (B\xi_\ell - \chi(\xi_\ell))\, \nabla_x K(x, x_\ell)\cdot n \right] dS_x + O((r_N)^4\, N), \qquad \text{at } \partial D_i(t) \tag{2.34}$$

Estimate (2.34) can be derived as follows. We decompose the last term on the right hand side of (2.33) in the average value in the particle centred at x_i plus a remainder term. The average value is the integral term in (2.34). Concerning the remainder, we can estimate it computing the highest order correction in Taylor's expansion for $\nabla_x K(x, x_\ell)$. Assuming that on average, particles remain at distances of order one, we would obtain that the second derivatives $\nabla_x^2 K(x, x_\ell)$ would be roughly of order one, and for radii of order r_N the estimate would follow.

Using the classical Gauss theorem we can rewrite the first term of (2.34) as:

$$-(r_N)^3 \int_{D_i(t)} \left[\sum_{\ell=1,\ell\neq i}^{N} (B\xi_\ell - \chi(\xi_\ell))\, K(x, x_\ell) \right] dx, \qquad i \neq \ell \tag{2.35}$$

A rather rough estimate of this term can be easily derived as follows. We have shown that in most configurations all the particles remain at least at distances of order $(\varepsilon_N)^{1/3} \approx r_N$. We can then estimate (2.35) as $O((r_N)^5\, N)$.

On the other hand, the possibly nonradial corrective term $O((r_N)^4\, N)$ in (2.34) is completely negligible if $\varepsilon_N N^2 \ll 1$.

On the other hand, using (2.25) the second term of the right hand side of (2.33) can be shown to be of order $O((r_N)^4)$, hence negligible.

It is interesting to remark that the estimates above hold also if the radius of the particles ξ_i increase, due to the presence of the term B that as will be seen later will be approximately like in (2.18). The consequence of this will the that terms like $B\xi_i$ would remain bounded by constants of order one.

Finally, we notice that in the last term in (2.32) we can make a similar approximation by the monopolar contribution, due to the fact that the boundary conditions near each particle are radial to the leading order. We then write:

$$-(r_N)^3\, N \frac{\partial V}{\partial n} \approx -\frac{(r_N)^3\, N}{|\partial D_i(t)|} \int_{\partial D_i(t)} \frac{\partial V}{\partial n}\, dS \tag{2.36}$$

Summarizing, to the leading order we have obtained that the corrections to the normal velocity at the surface of each particle are radial. Using

(2.32), (2.33), and (2.36) we then obtain the following equation for the radius of each particle:

$$\frac{d\xi_i}{dt} = -\frac{(1 - B(t)\,\xi_i)}{(\xi_i)^2} - |\Omega|\,\varepsilon_N \eta_i \tag{2.37}$$

where:

$$\eta_i = \frac{1}{|\partial D_i(t)|} \int_{\partial D_i(t)} \frac{\partial V}{\partial n}\, dS \tag{2.38}$$

and where $B(t)$ has to be determined using (2.17). The function V solves (2.28) with boundary condition (2.31). If we formally set $\varepsilon_N = 0$ in (2.38), we would recover the usual LSW model. Notice that determining η_i we have to take into account the probabilistic character of the positions of the particles. Comparing (2.38) and (2.22) it readily follows that $\sigma_i(t) = -|\Omega|\,\varepsilon_N \eta_i(t)$. Our next goal is to simplify the form of $\sigma_i(t)$ by taking into account that the number of particles N is very large.

It would be interesting to understand if the estimates that have been made above are independent on the evolution of the system for $t > 0$. More precisely, it has been assumed that, initially, the positions of the particles, as well as the distribution of radii are independent variables. However, during the evolution of the system, correlations could develop and this fact could affect seriously the evolution of the distribution of radii. Some of the previous estimates are independent of the distribution of particles (for instance the first three terms on the right hand side of (2.29)). Other estimates use some kind of averaging in the distribution of the particles (cf. for instance the last term in (2.29) or (2.35)). However, it is important to point out that in these last cases, we are deriving upper bounds on the size of terms involved. In particular, this implies that after the vanishing of several particles, the estimates that have been derived for the initial distribution of particles could only improve. Some effects, for instance drifting of particles or higher order corrections, have been neglected because they are produced by terms smaller than the ones that have been kept. It would be unlikely that these higher order corrections could change drastically the dynamics of the problem, although a more rigorous and detailed analysis of these higher order terms could be convenient.

2.2. Computing the Corrective Term $\sigma_i(t)$

We remark that the function η_i depends of the coordinates x_ℓ, ξ_ℓ through (2.28), (2.31), and (2.38). The radii ξ_ℓ are functions of t, but the positions x_ℓ remain constant in time according to the approximations made in the derivation of the model. If dipolar terms were taken into account in the computation of V_n some slow drifting of the particles would take place. However, within the range of accuracy used to obtain (2.37), it is natural to ignore translations of particles. We will assume henceforth that the coordinates x_ℓ and ξ_ℓ are independent random variables uniformly distributed in the domain Ω according to the probability distribution (2.21).

We recall that the distribution of radii evolves according to the system (2.37). Our main goal in this Section is to compute a simpler approximation of the random variables η_i. To this end, we need as a preliminary step an approximation of the solutions of the problem (2.28), (2.31).

Since for most of the configurations of the particles the distance between then is much larger than r_N, it is natural to assume as in (2.26) the following approximation for V:

$$V \approx \sum_{\ell=1}^{N} [\alpha_\ell K(x, x_\ell) + B_\ell] \tag{2.39}$$

where the constants α_ℓ, β_ℓ have to be determined using the boundary conditions (2.31). Near the ith particle the leading term in (2.39) is $\alpha_i K(x, x_i) + \beta_i$, and using (2.25) it follows that this term can be approximated as $(\alpha_i/|x_i - x_\ell|) + \beta_i$. Let us assume that this term agrees with the constant k_i in (2.31) at the boundary. It then follows that $\alpha_i = (k_i - \beta_i)\, r_N \xi_i$. We can then rewrite the boundary condition (2.31) to the leading order as the following system of equations for the coefficients β_ℓ:

$$\sum_{\ell=1,\, \ell \neq i}^{N} [(k_\ell - \beta_\ell)\, r_N \xi_\ell K(x_i, x_\ell) + \beta_\ell] + (k_i - \beta_i)\, O((r_N)^2) = 0, \qquad i = 1,..., N \tag{2.40}$$

where as usual in this paper only monopolar terms have been kept, and we have used again (2.25).

In the limit case $\varepsilon_N N^2 \ll 1$, the terms with the form $(\beta_\ell r_N \xi_\ell)/|x_i - x_\ell|$ can be neglected, as well as the contribution due to the terms $(k_i - \beta_i)\, O((r_N)^2)$ that is even smaller. To check this, notice that under this assumption (2.40) becomes:

$$\sum_{\ell=1,\, \ell \neq i}^{N} \beta_\ell = - \sum_{\ell=1,\, \ell \neq i}^{N} \frac{k_\ell r_N \xi_\ell}{|x_i - x_\ell|} \equiv \sigma_i, \qquad i = 1,..., N \tag{2.41}$$

The solution of (2.41) can be written in the form:

$$\beta_i = S - \sigma_i, \qquad i = 1,..., N \tag{2.42}$$

where:

$$S \equiv \sum_{\ell=1}^{N} \beta_\ell = \frac{1}{(N-1)} \sum_{\ell=1}^{N} \sigma_\ell \tag{2.43}$$

The contribution of the terms of the form $(\beta_\ell r_N \xi_\ell)/|x_i - x_\ell|$ to the right hand side of (2.41) can be estimated as follows:

$$\left| \sum_{\ell=1,\, \ell \neq i}^{N} \left[\frac{\beta_\ell r_N \xi_\ell}{|x_i - x_\ell|} \right] \right| \leqslant C\, |\sigma|\, N r_N \int_\Omega \frac{dy}{|x-y|} \tag{2.44}$$

where $|\sigma|$ denotes the order of magnitude of the coefficients σ_i in (2.41). Under our assumptions on ε_N we have $Nr_N \ll 1$, whence the contribution due to this corrective term would be negligible. The last term on the left hand side of (2.40) is even smaller. It is not hard to make this argument rigorous by using a perturbative series.

In any case, we can suppose to the leading order that the coefficients β_i are given by (2.42), (2.43). Using then (2.39), as well as the previously derived approximation for the coefficients α_i, we obtain the following value for V:

$$V \approx \sum_{\ell=1}^{N} \left[(k_\ell - \beta_\ell)\, r_N \xi_\ell K(x, x_\ell) + \beta_\ell \right] \tag{2.45}$$

where the β_ℓ are as in (2.42), (2.43). Taking into account (2.38) we arrive at the following approximation for η_i:

$$\eta_i \approx \frac{\beta_i - k_i}{r_N \xi_i}, \qquad i = 1,..., N \tag{2.46}$$

We can obtain a further simplification of (2.46). Let us denote as $|k|$ the order of magnitude of the coefficients k_i. Taking into account (2.41), and arguing as in the derivation of (2.44), we can derive the following estimate for $|\sigma|$:

$$|\sigma| \leqslant CNr_N\, |k|$$

and from (2.42) and (2.43) we then deduce that:

$$|\beta_i| \leqslant CNr_N\, |k| \tag{2.47}$$

The estimate (2.47) implies that, in the limit $\varepsilon_N N^2 \ll 1$, the contribution due to the term β_i is negligible compared with k_i. This provides the approximation:

$$\eta_i \approx -\frac{k_i}{\xi_i}, \qquad i=1,\dots, N \tag{2.48}$$

that will be used from now on. The coefficients k_i are the ones given in (2.31). Summarizing, to the leading order (2.37) can then be rewritten as:

$$\frac{d\xi_i}{dt} = -\frac{(1-B(t)\,\xi_i)}{(\xi_i)^2} + \frac{|\Omega|}{\xi_i}\left(\frac{\varepsilon_N}{r_N}\right) k_i, \qquad i=1,\dots, N \tag{2.49}$$

In view of the analysis above, we have reduced our problem to the study of the random variables k_i that have a more explicit expression than the η_i's (cf. (2.31)). As a next step we intend to estimate the terms k_i. Notice that the functions k_i depend on t, since their value depend on the positions of the particles x_ℓ, and some of them disappear as time runs. In principle this could generate some "noisy" variation of $k_i(t)$, that could be relevant near the region of largest maxima. One of the main goals of this paper is to show that this random variation of $k_i(t)$ is not strong enough to stabilise the distribution of particles to the LSW solution.

By assumption, a particle that disappears in our model cannot be generated again. This implies that $N(t) = \sum_{m=1}^{N} \chi(\xi_m(t))$ is monotonically decreasing. Notice that we can write:

$$k_i(t) = \frac{1}{N} \sum_{m=1}^{N} \left(\chi(\xi_m(t)) - \frac{N(t)\,\xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)}\right) M_{i,m}, \qquad i=1,\dots, N \tag{2.50}$$

where by notational simplicity, we define:

$$M_{\ell,m} = K(x_\ell, x_m), \qquad \ell \neq m \tag{2.51}$$

$$M_{\ell,m} = 0, \qquad \ell = m \tag{2.52}$$

By assumption $\xi_m(t) = 0$ after extinction of the m-particle takes place. Combining (2.49) with the volume conservation (2.17) we readily obtain:

$$B(t) = \frac{N(t) - |\Omega|\,(\varepsilon_N/r_N) \sum_{i=1}^{N} k_i(t)\,\xi_i(t)}{\sum_{i=1}^{N} \xi_i(t)} \tag{2.53}$$

Plugging (2.53) into (2.49) and using also (2.50) we arrive at:

$$\frac{d\xi_i}{dt} = -\frac{1}{(\xi_i)^2} + \frac{N(t)}{\sum_{x=1}^{N} \xi_s(t)} \frac{1}{\xi_i} + \frac{N(t)}{\sum_{s=1}^{N} \xi_s(t)} \frac{\lambda_i(t)}{\xi_i} \tag{2.54}$$

where:

$$\lambda_i(t) \equiv \frac{|\Omega|\, \varepsilon_N}{r_N N(t)} \sum_{m=1}^{N} \frac{1}{N} \sum_{\ell=1}^{N} \left[\left(\chi(\xi_m(t)) - \frac{N(t)\, \xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)} \right) (M_{i,m} - M_{\ell,m}) \right] \xi_\ell(t) \tag{2.55}$$

The model that will be considered in this paper is (2.54) and (2.55) with a random distribution of initial radii whose choice will be described in detail in next Section. On the other hand the coefficients $M_{i,m}$ are random variables defined by means of (2.51), (2.52). Notice that the volume conservation (2.17) immediately holds by construction. In Section 4 some estimates will be derived showing that the functions $\lambda_i(t)$ are simultaneously small for all the particles, with a probability that approaches one if the number of particles approaches infinity.

Corrective terms for the LSW dynamics closely related to (2.55) have been recently obtained in ref. 1, where the problem of computing higher order corrections to the LSW mean field theory has been addressed. Actually, the authors of ref. 1 have computed also higher order terms that are responsible for the drifting of particles, and they have also estimated the effect of the corrective terms in particular configurations of particles.

3. STOCHASTIC PROPERTIES OF THE INITIAL DISTRIBUTION OF RADII

We now describe the initial distribution of radii that will be used to solve the model (2.54), (2.55). To this end it is more convenient to work with distributions of probability instead than with probability densities. Given any continuous increasing function F satisfying $F(0)=0$, $F(\infty)=1$, we will assign the initial density of radii $\xi_s(0)$, $s=1,..., N$ as a set of independent random variables with a probability:

$$P(\xi_s(0) \leqslant \xi) = F(\xi) \tag{3.1}$$

The continuity assumption on F is not actually necessary on mathematical grounds, but it will be needed for our purposes here. By convenience, we will assume also that $1-F$ is compactly supported.

Before stating more precise results, it is instructive to get some heuristic understanding on the distribution function associated to the discrete distribution of particles. To this end, we define the discrete distribution function as:

$$F_N(\xi) = \frac{1}{N} \sum_{i=1}^{N} \chi(\xi - \xi_i(0)) \tag{3.2}$$

where, as indicated in Section 2, $\chi(s) = 1$ if $s > 0$ and $\chi(0) = 0$.

It is trivially seen that $F_N(\xi)$ is just the number of particles in the distribution $\xi_s(0)$, $s = 1,..., N$ with a radius smaller or equal to ξ. Using the classical large numbers law, as well as the continuity assumption on F, it is not hard to check that with probability one:

$$\lim_{N \to \infty} F_N(\xi) = F(\xi) \tag{3.3}$$

uniformly on ξ (cf. for instance ref. 4). We can obtain a better intuition on the effect of the fluctuations computing the higher order correction in the limit (3.3). Let us write:

$$F_N(\xi) = F(\xi) + \frac{1}{\sqrt{N}} n(\xi) \tag{3.4}$$

where by definition:

$$n(\xi) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\chi(\xi - \xi_i) - \langle \chi(\xi - \xi_i) \rangle) \tag{3.5}$$

and where from now on $\langle \cdot \rangle$ denotes average with respect to the measure of probability:

$$dv(\xi_s(0), s = 1,..., N) = \prod_{s=1}^{N} dF(\xi_s(0)) \tag{3.6}$$

The function $n(\xi)$ is a stochastic process that can be analysed in the limit $N \to \infty$ by standard methods. More precisely, let us fix a finite set of different real numbers $\bar{\xi}_k$, $k = 0,..., L$. The characteristic function of the stochastic variables $n(\bar{\xi}_1), n(\bar{\xi}_2),..., n(\bar{\xi}_L)$ can be computed in the limit $N \to \infty$ as follows:

$$
\begin{aligned}
\lim_{N\to\infty} & \left\langle \exp\left(i \sum_{\ell=1}^{L} n(\bar{\xi}_\ell)\,\theta_\ell\right)\right\rangle \\
&= \lim_{N\to\infty} \left\langle \prod_{i=1}^{N} \left(1-\frac{1}{2N}\left(\sum_{\ell=1}^{L} (\chi(\bar{\xi}_\ell-\xi_i)-\langle\chi(\bar{\xi}_\ell-\xi_i)\rangle)\,\theta_\ell\right)^2\right)\right\rangle \\
&= \exp\left(-\frac{1}{2}\left\langle\left(\sum_{\ell=1}^{L} (\chi(\bar{\xi}_\ell-\xi_1)-\langle\chi(\bar{\xi}_\ell-\xi_1)\rangle)\,\theta_\ell\right)^2\right\rangle\right) \\
&= \exp\left(-\frac{1}{2}\sum_{\ell=1}^{L}\sum_{k=1}^{L} (\min(F(\bar{\xi}_\ell), F(\bar{\xi}_k))-F(\bar{\xi}_\ell)\cdot F(\bar{\xi}_k))\,\theta_\ell\theta_k\right) \\
&\equiv J(\theta)
\end{aligned}
\tag{3.7}
$$

The characteristic function $J(\theta)$ in (3.7) is well known, and can be easily written in terms of the stochastic process known as brownian bridge. Let us denote as $w(t)$ the classical brownian motion, or Wiener process. The brownian bridge, defined for $t \in [0, 1]$ is the stochastic process $b(t) = w(t) - tw(1)$. Its characteristic function is given by:

$$
\begin{aligned}
&\left\langle \exp\left(i \sum_{\ell=1}^{N} (w(t_\ell)-t_\ell w(1))\,\theta_\ell\right)\right\rangle \\
&\quad = \exp\left(-\tfrac{1}{2}\left[\sum_{\ell=1}^{N}\sum_{j=1}^{N} \min\{t_\ell, t_j\}\,\theta_\ell\theta_j - \left(\sum_{\ell=1}^{N} t_\ell\theta_\ell\right)^2\right]\right)
\end{aligned}
\tag{3.8}
$$

as can be immediately checked using the characteristic function for the brownian motion (cf. for instance ref. 10). Comparing (3.7) and (3.8), we then immediately obtain the following formula for $n(t)$ in (3.5):

$$
n(t) = b(F(\xi)) \tag{3.9}
$$

where $b(\cdot)$ is the brownian bridge. As a matter of fact, the analysis made above is a rather classical one in the theory of empirical statistical distributions. A more detailed and rigorous analysis of the derivation of (3.9) could be found in ref. 12.

Summarizing, in the limit of large numbers of particles, $F_N(\xi)$ in (3.4) can be approximated by using (3.9). There is an aspect of this approximation, however that could be a bit misleading and is worth clarifying. By definition the function $F_N(\xi)$ in (3.4) is monotonically increasing. If we just use the approximation (3.9) in a naive way in (3.4), it would then follow that $\tau + (b(\tau)/\sqrt{N})$ would be an increasing function, whence in particular, using the symmetry properties associated to the brownian motion, it would follow that $b(\tau)$ would be Lipschitz continuous at any point, but it is well known that the paths associated to the brownian motion are not differentiable anywhere point. The explanation of this apparent paradox is just that the Lipschitz constant that would be obtained for $b(\tau)$ grows as $\sqrt{N}$ as $N \to \infty$. It is well known that the paths associated to the brownian motion behave locally, near each point $\tau = \tau_0$ as $(\tau - \tau_0)^{1/2}$, with some logarithmic corrections that we ignore for the moment. On the other hand, notice that the increments on $\tau - \tau_0$ take place in discrete amounts of order $1/N$. In particular, the computations in (3.7) are not admissible for such small variations of τ. In another way, the approximation (3.9) is valid only for distances on τ larger than $1/N$. The variations that $b(\tau)$ would suffer in such distances are of order $\sqrt{1/N}$, a result that agrees with the Lipschitz constant previously computed. If one tries to compute variations of $F_N(\xi)$ at distances of order $1/N$ the approximation (3.9) is not valid anymore, and the discrete character of function $F_N(\xi)$ enters into the problem. However, this is not a serious difficulty in our approach, since we are interested in continuum limits of particles, whence we will always consider increments on τ much larger than $1/N$.

We now proceed to obtain some precise estimates on the fluctuations on the number of particles that takes place if we examine "macroscopic scales" in the space of radii. To this end, for each N let us introduce a number R_N satisfying:

$$\log(N) \ll R_N \ll N \tag{3.10}$$

Our goal is to decompose the space of radii ξ in a set of intervals in such a way that the expected number of particles in each subinterval is R_N. The choice (3.10) would ensure that the number of particles on each subinterval allows to consider them as a "macroscopic" ensemble. Let us suppose that function $F(\xi)$ is supported in the interval $[0, \bar{\xi}_0]$. We then define a sequence of numbers $\bar{\xi}_k$ by means of the formula:

$$1 - F(\bar{\xi}_k) = \frac{R_N}{N} k \tag{3.11}$$

Notice that in this form the interval $[0, \bar{\xi}_0]$ is splitted in a family of intervals whose interiors do not intersect, namely:

$$[0, \bar{\xi}_0] = \bigcup_{k=1}^{M_N} [\bar{\xi}_k, \bar{\xi}_{k-1}] \tag{3.12}$$

where $M_N = N/R_N$. By convenience we will assume, without great loss of generality, that M_N is an integer number.

By assumption the values of the initial distribution of radii are assigned according to the probability distribution (3.6). We want to determine the number of initial radii that fall in each subinterval $[\bar{\xi}_k, \bar{\xi}_{k-1}]$. Actually such a number is given by the stochastic variable:

$$Y_k \equiv \sum_{j=1}^{N} \chi_{[\bar{\xi}_k, \bar{\xi}_{k-1}]}(\xi_j(0)) \tag{3.13}$$

where $\chi_{[\bar{\xi}_k, \bar{\xi}_{k-1}]}$ denotes the characteristic function of the corresponding subinterval. Using (3.6), (3.11), (3.13) it readily follows that the average number of particles in each subinterval is:

$$\langle Y_k \rangle = R_N \tag{3.14}$$

We then have that the average number of particles for each subinterval is R_N. In particular, this implies that in a "coarse-grained" or "macroscopic" scale the discrete distribution $F_N(\xi)$ defined in (3.2) is in average rather close to $F(\xi)$. Our goal is to show that under suitable assumptions, $F_N(\xi)$ is close to $F(\xi)$ in a pointwise sense. To this end we show the following result:

Proposition 3.1. Let us assume that (3.10) holds. There exists $\Gamma > 0$ large enough, independent on N, such that the following inequality is satisfied in a set whose probability approaches to one as $N \to \infty$:

$$\max_{k=1,\dots,M_N} |Y_k - R_N| \leqslant \Gamma \sqrt{R_N \log(N)} \tag{3.15}$$

Proof. The stochastic variable Y_k can be considered as the number of realisations of the outcome to "fall in the interval $[\bar{\xi}_k, \bar{\xi}_{-1}]$" among N independent trials of the stochastic variable $\xi_j(0)$, whose probability distribution is given by (3.6). Since the probability of falling in the interval $[\bar{\xi}_k, \bar{\xi}_{k-1}]$ is R_N/N for every value k, it easily follows that Y_k is a multinomial variable, and the probability of achieving the result $\{Y_k = n_k, k = 1,\dots, M_N\}$ where $\sum_{k=1}^{M_N} n_k = N$ is:

$$p\{Y_k = n_k, k = 1,\dots, M_N\} = \frac{(N)!}{\prod_{k=1}^{M_N} (n_k)!} \left(\frac{R_N}{N}\right)^N \tag{3.16}$$

Let us pick $A = R_N - \alpha$, $B = R_N + \beta$, integer numbers, where:

$$\min\{\alpha, \beta\} = [\Gamma \sqrt{R_N \log(N)}] \tag{3.17}$$

Γ is a fixed constant to be precised later, and $[\cdot]$ stands by the integer part of a number. Notice that in order to prove (3.15) it is enough to show that the probability of the outcome:

$$S=\{A \leqslant Y_k \leqslant B, k=1,..., M_N\}$$

approaches to one as $N \to \infty$, and taking into account (3.16), it follows that:

$$\begin{aligned} p(S) &= \sum_{n_1=A}^{B} \sum_{n_2=A}^{B} \cdots \left(\sum_{k=1}^{M_N} n_k = N \right) \cdots \sum_{n_{M_N}=A}^{B} \frac{(N)!}{\prod_{k=1}^{M_N} (n_k)!} \left(\frac{R_N}{N} \right)^N \\ &\equiv \sigma_N(A, B) \left(\frac{R_N}{N} \right)^N \end{aligned} \tag{3.18}$$

We can compute explicitly $\sigma_N(A, B)$ by means of a generatrix function. Indeed, let us define:

$$G(z; A, B) = \sum_{N=0}^{\infty} \sigma_N(A, B)\, z^N \tag{3.19}$$

Using that $\sum_{k=1}^{M_N} n_k = N$, as well as the fact that

$$\left(\sum_{n=A}^{B} \frac{z^n}{(n)!} \right)^{M_N} + \sum_{n_1=A}^{B} \sum_{n_2=A}^{B} \cdots \sum_{n_{M_N}=A}^{B} \frac{z^{n_1+n_2+ \cdots n_{M_N}}}{\prod_{k=1}^{M_N} (n_k)!}$$

it readily follows that $G(z; A, B)$ may be written as:

$$G(z; A, B) = (N)! \left(\sum_{n=2}^{B} \frac{z^n}{(n)!} \right)^{M_N} \tag{3.20}$$

The classical Cauchy formula for the coefficients of a power series yields:

$$\sigma_N(A, B) = \frac{(N)!}{2\pi i} \int_{\gamma} \frac{(\sum_{n=A}^{B} (\zeta^n/(n)!))^{M_N}}{(\zeta)^{N+1}}\, d\zeta \tag{3.21}$$

where γ is any contour surrounding the origin of coordinates. In order to compute the asymptotics of $\sigma_N(A, B)$ as $N \to \infty$ we use the classical

Laplace method. To this end, notice that by the definition of M_N we can rewrite (3.21) as:

$$\sigma_N(A, B) = \frac{(N)!}{2\pi i} \int_{\gamma} (P(\zeta))^{M_N} \frac{d\zeta}{\zeta} \tag{3.22}$$

where:

$$P(\zeta) = \sum_{n=A}^{B} \frac{\zeta^{n-R_N}}{(n)!} \tag{3.23}$$

The function $P(\zeta)$ is convex in the line $\zeta > 0$. Moreover, using the fact that $A < R_N < B$, it readily follows that there exists a unique value $\hat{\zeta}_N = \hat{\zeta}_N(A, B)$ such that:

$$P'(\hat{\zeta}_N) = 0 \tag{3.24}$$

Deforming the contour if needed, we can assume that γ in (3.22) is a circle centered at the origin with radius $\hat{\zeta}_N$. By (3.10), it follows that $M_N \gg 1$ as $N \to \infty$. If we write $\zeta = |\zeta|\, e^{i\theta}$ in (3.23) we can easily obtain that $|P(\zeta)| < P(\hat{\zeta}_N)$ if $\theta \neq 0$, whence the contribution in (3.22) from the region away from the point $\zeta = \hat{\zeta}_N$ is negligible. Writing $(P(\zeta))^{M_N} = \exp(M_N \ln(P(\zeta)))$, and expanding by means of Taylor's theorem $P(\zeta)$ in a neighbourhood of the point $\zeta = \hat{\zeta}_N$, we arrive, after some elementary computations at:

$$\sigma_N(A, B) = \frac{(N)!\, (P(\hat{\zeta}_N))^{M_N}}{2\pi i \hat{\zeta}_N} (1 + o(1)) \int_{\hat{\zeta}_N + i\mathbb{R}} \exp\left(\frac{M_N P''(\hat{\zeta}_N)}{2P(\hat{\zeta}_N)} (\zeta - \hat{\zeta}_N)^2 \right) d\zeta$$

as $N \to \infty$. Introducing then the change of variables $\zeta = \hat{\zeta}_N + \sqrt{2P(\hat{\zeta}_N)/M_N P''(\hat{\zeta}_N)}\, i\xi$, we obtain the approximation:

$$\sigma_N(A, B) = \frac{(N)!\, (P(\hat{\zeta}_N))^{M_N}}{2\sqrt{\pi}\, \hat{\zeta}_N} (1 + o(1)) \sqrt{\frac{2P(\hat{\zeta}_N)}{M_N P''(\hat{\zeta}_N)}} \quad \text{as} \quad N \to \infty \tag{3.25}$$

As a next step we proceed to approximate $\hat{\zeta}_N$ in (3.25), taking into account its very definition in (3.24). Using (3.23), we easily obtain after some simple manipulations that:

$$P'(\zeta) = \sum_{n=A}^{B} \frac{\zeta^{n-R_N-1}}{(n-1)!} - R_N \sum_{n=A}^{B} \frac{\zeta^{n-R_N-1}}{(n)!} \frac{\zeta^{A-R_N-1}}{(A-1)!}$$
$$- \frac{\zeta^{B-R_N}}{(B)!} + \left(1 - \frac{R_N}{\zeta}\right) P(\zeta)$$

whence Eq. (3.24) becomes:

$$P(\zeta) = \frac{1}{\zeta - R_N}\left[\frac{\zeta^{B-R_N+1}}{(B)!} - \frac{\zeta^{A-R_N}}{(A-1)!}\right] \tag{3.26}$$

Let us define a new variable x by means of

$$\zeta = R_N(x+1) \tag{3.27}$$

Using this new variable, as well as the definition of $P(\zeta)$ in (3.23), (3.26) can be rewritten as:

$$x = \frac{H(x)}{(\sum_{n=A}^{B} (\zeta^n/(n)!))} \tag{3.28}$$

where:

$$H(x) \equiv \left[\frac{R^B(1+x)^{B+1}}{(B)!} - \frac{R^{A-1}(1+x)^A}{(A-1)!}\right] \tag{3.29}$$

We have already seen that (3.28) has a unique solution in the region $x > -1$. As a matter of fact we will show that such a solution lies in a region where the following both inequalities are satisfied:

$$(A+x)^{B+1} = 1 + o(1) \tag{3.30}$$

$$(1+x)^A = 1 + o(1) \tag{3.31}$$

as $N \to \infty$. If (3.30) and (3.31) hold, then:

$$\sum_{n=A}^{B} \frac{\zeta^n}{(n)!} = (1+o(1)) \sum_{n=A}^{B} \frac{(R_N)^n}{(n)!}$$

as $N \to \infty$. We can then rewrite (3.28) as:

$$x = \frac{(1+o(1))}{\sum_{n=A}^{B} ((R_N)^n/(n)!)}\left[\frac{(R_N)^B}{(B)!} - \frac{(R_N)^{A-1}}{(A-1)!}\right] + o(1)\,\frac{\max\{(R_N)^B/(B)!, (R_N)^{A-1}/(A-1)!\}}{\sum_{n=A}^{B} ((R_N)^n/(n)!)}$$

as $N \to \infty$. Making the change of variables $n = R_N + \ell$, and using Stirling's formula as well as the definition of A, B, we arrive after some computations at:

$$x = \frac{(1+o(1))}{\sum_{\ell=-\alpha}^{\beta} (e^{\ell}/(1+\ell/R_N)^{R_N+\ell+1/2})} \times \left[\frac{e^{\beta}}{(1+\beta/R_N)^{R_N+\beta+1/2}} - \frac{e^{-(\alpha+1)}}{(1-(\alpha+1)/R_N)^{R_N-\alpha-1/2}} \right] + o(1) \frac{\left[\begin{array}{l} \max\{(e^{\beta}/(1+(\beta/r_N))^{R_N+\beta+1/2}), \\ \quad (e^{-(\alpha+1)}/(1-((\alpha+1)/R_N))^{R_N-\alpha-1/2})\} \end{array} \right]}{\sum_{\ell=-\alpha}^{\beta} (e^{\ell}/(1+\ell/R_N)^{R_N+\ell+1/2})} \tag{3.32}$$

Using now the fact that $\log(N) \ll R_N$ (cf. (3.10)), as well as (3.17) we can rewrite (3.32) as:

$$x = \frac{(1+o(1))}{\sum_{\ell=-\alpha}^{\beta} e^{\ell(R_N+\ell)\log(1+\ell/R_N)}} \times [e^{\beta-(R_N+\beta)\log(1+\beta/R_N)} - e^{-(\alpha+1)-(R_N-(\alpha+1))\log((1-(\alpha+1)/R_N))}] + o(1) \frac{\left[\begin{array}{l} \max\{e^{\beta-(R_N+\beta)\log(1+\beta/R_N)}, \\ \quad e^{-(\alpha+1)-(R_N-(\alpha+1))\log((1-(\alpha+1)/R_N))}\} \end{array} \right]}{\sum_{\ell=-\alpha}^{\beta} e^{\ell-(R_N+\ell)\log(1+\ell/R_N)}} \tag{3.33}$$

Taking into account that for $|\ell| \leqslant \Gamma\,|\log(N)|$ there holds $\ell-(R_N+\ell) \times \log(1+\ell/R_N) = -(1/2)(\ell^2/R_N) + O(\ell^3/(R_N)^2)$, it is not hard to approximate the sums in (3.33) using (3.10). We then eventually arrive at:

$$x = \frac{(e^{-(\beta)^2/2R_N} - e^{-(\alpha)^2/2R_N})}{\sqrt{2\pi R_N}} + o(1)\max\left\{\frac{e^{-(\beta)^2/2R_N}}{\sqrt{R_N}}, \frac{e^{-(\alpha)^2/2R_N}}{\sqrt{R_N}}\right\} \tag{3.34}$$

where we have used (3.17) as well as the well known fact that $\int_{-\infty}^{\infty} e^{-z^2/2}\,dZ = \sqrt{2\pi}$. Formula (3.34) provides an asymptotic expression for the solution of (3.28). It only remains to check that (3.30) and (3.31) holds. This immediately follows from the fact that (3.17), (3.34) would provide an estimate for x of the form

$$|x| = O\left(\frac{e^{-((\Gamma)^2/2)\log(N)}}{\sqrt{R_N}}\right) = O\left(\frac{1}{\sqrt{R_N}\,(N)^{(\Gamma)^2/2}}\right) \quad \text{as} \quad N \to \infty \tag{3.35}$$

Notice that (3.30) and (3.31) are satisfied if $B\,|x| \ll 1$. Since $B \leqslant 2R_N$ for large values of N, it is enough to have $\sqrt{R_N}/(N)^{(\Gamma)^2/2} \ll 1$, and by (3.10), we would derive this estimate as $N \to \infty$ if we just choose $\Gamma > 1$. As a matter of fact, we will need to choose Γ satisfying:

$$\Gamma > \sqrt{2} \tag{3.36}$$

As a final step, we use (3.34) in order to derive an asymptotic formula for $\sigma_N(A, B)$ by means of (3.25). Using (3.23) and (3.27) as well as Stirling's formula in (3.25) we obtain:

$$\sigma_N(A, B) = (N)^{N+1/2}\, e^{-N} \left(\frac{R_N}{N}\right)^{1/2} \frac{\left(\sum_{n=A}^{B} [R_N^{n-R_N}(1+x)^{n-R_N}]/(n)!\right)^{N/R_N}}{R_N} \times \sqrt{\frac{P(\hat{\zeta}_N)}{P''(\hat{\zeta}_N)}}\,(1+o(1)) \tag{3.37}$$

Combining (3.35) and (3.36), it easily follows that $(1+x)^{n-R_N} = 1+o(1)$ as $N \to \infty$. Using this fact as well as (3.18), we derive from (3.37):

$$p(S) = \frac{e^{-N}}{\sqrt{R_N}} \left(\sum_{n=A}^{B} \frac{R_N^n}{(n)!}\right)^{N/R_N} \sqrt{\frac{P(\hat{\zeta}_N)}{P''(\hat{\zeta}_N)}}\,(1+o(1)) \tag{3.38}$$

Finally, notice that by (3.23), (3.35) and (3.36):

$$\frac{P(\hat{\zeta}_N)}{P''(\hat{\zeta}_N)} = \frac{\sum_{n=A}^{B} R_N^n/(n)!}{\sum_{n=A}^{B} ((n-R_N)(n-R_N-1)\, R_N^n)/(n)!}\,(1+o(1))$$

as $N \to \infty$. Using Stirling's formula again, making the change of variables $n = R_N + \ell$, and approximating the sums by integrals as above, we derive after some elementary computations that:

$$\frac{P(\hat{\zeta}_N)}{P''(\hat{\zeta}_N)} = R_N(1+o(1))$$

Plugging this expression into (3.38) we obtain:

$$p(S) = e^{-N} \left(\sum_{n=A}^{B} \frac{R_N^n}{(n)!}\right)^{N/R_N} (1+o(1)) \tag{3.39}$$

The limit of the right hand side of (3.39) can be easily computed in the following way. Let us write:

$$e^{-N}\left(\sum_{n=A}^{B}\frac{R_N^n}{(n)!}\right)^{N/R_N}=\left(1-e^{-R_N}\sum_{n=0}^{A-1}\frac{R_N^n}{(n)!}-e^{R_N}\sum_{n=B+1}^{\infty}\frac{R_N^n}{(n)!}\right)^{N/R_N} \tag{3.40}$$

Notice that:

$$\begin{aligned} e^{-R_N}\sum_{n=0}^{A-1}\frac{R_N^n}{(n)!} &\leqslant Ce^{-R_N}\left(1+\sum_{n=1}^{A-1}\frac{R_N^n e^n}{(n)^{n+1/2}}\right)\\ &\leqslant C(e^{-R_N}+\sqrt{R_N}\, e^{-\alpha-(R_N-\alpha)\log(1-\alpha/R_N)})\\ &\leqslant C\sqrt{R_N}\, e^{-(\Gamma^2/2)\log(N)}=\frac{C\sqrt{R_N}}{(N)^{\Gamma^2/2}} \end{aligned}$$

An estimate of the last series in (3.40) can be obtained in an analogous way. Using then (3.36), it readily follows that the right hand side of (3.40) approaches to one, whence:

$$p(S)=1-o(1)$$

as $N\to\infty$. This concludes the proof of Proposition 3.1. ▮

4. PROBABILISTIC ESTIMATES OF THE SPATIAL FLUCTUATIONS TERMS

We now proceed to obtain that the stochastic terms $\lambda_i(t)$ in (2.55) are small with probability close to one if the number of particles is large. The main result of this section is the following:

Proposition 4.1. There exist constants $C>0$ and $L>0$ depending only on Ω, such that, if the particles x_j are uniformly distributed in Ω, then for any $\theta>0$, there holds:

$$\sup_{i=1,\dots,N}|\lambda_i(t)|\leqslant C(\varepsilon_N)^{2/3}(N)^{1/3}(\theta+1)\sup_{i=1,\dots,N}\xi_i(t) \tag{4.1}$$

with a probability p satisfying:

$$p\geqslant 1-\frac{L}{(\theta)^2} \tag{4.2}$$

Notice that choosing θ large enough in (4.2), (4.1) states that with a probability close to one, all the perturbative terms $\lambda_i(t)$ can be made uniformly small in the precise form indicated in Proposition 4.1. Actually, under the assumption $\varepsilon_N \ll 1/N^2$, (4.1) will be enough for our purposes.

Proof of Proposition 4.1. From the definition of the terms $\lambda_i(t)$ in (2.55), using the nonnegativity of the coefficients $M_{i,m}$ as well as (2.7) we easily obtain that:

$$|\lambda_i(t)| \leqslant J_1 + J_2 \tag{4.3}$$

where:

$$J_1 = \frac{C(\varepsilon_N)^{2/3}(N)^{1/3}}{N(t)} \left(\max_{m=1,\dots,N} \frac{1}{N} \sum_{\ell=1}^{N} M_{\ell,m} \xi_\ell(t) \right) \times \left(\sum_{m=1}^{N} \left| \chi(\xi_m(t)) - \frac{N(t)\,\xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)} \right| \right) \tag{4.4}$$

and:

$$J_2 = C(\varepsilon_N)^{2/3}(N)^{1/3} \left(\frac{1}{N(t)} \sum_{\ell=1}^{N} \xi_\ell(t) \right) \times \left(\max_{i=1,\dots,N} \frac{1}{N} \sum_{m=1}^{N} M_{i,m} \left| \chi(\xi_m(t)) - \frac{N(t)\,\xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)} \right| \right) \tag{4.5}$$

We begin by estimating J_1. Notice that:

$$\begin{aligned} &\max_{m=1,\dots,N} \left(\frac{1}{N} \sum_{\ell=1}^{N} M_{\ell,m} \xi_\ell(t) \right) \\ &\qquad \leqslant \sup_{i=1,\dots,N} \xi_i(t) \cdot \max_{m=1,\dots,N} \left(\frac{1}{N} \sum_{\ell=1}^{N} M_{\ell,m} \right) \\ &\qquad = \sup_{i=1,\dots,N} \xi_i(t) \cdot \max_{m=1,\dots,N} \left(\frac{1}{N} \sum_{\ell=1,\ell\neq m}^{N} K(x_\ell, x_m) \right) \end{aligned} \tag{4.6}$$

where $K(x, y)$ is as in (2.24) and (2.25).

In order to estimate the term $\max_{m=1,\dots,N}((1/N) \sum_{\ell=1,\ell\neq m}^{N} K(x_\ell, x_m))$, we argue as follows. Let us define the function $f_m(x) = (1/N) \sum_{\ell=1,\ell\neq m}^{N} K(x_\ell, x_m)$. Notice that

$$\begin{aligned} \langle f_m(x) \rangle &= \int_{\Omega^N} f_m(x)\, dx_1 \cdots dx_N \\ &= \frac{(N-1)}{N} \int_{\Omega} K(\xi, 0)\, d\xi \equiv \frac{(N-1)\,\kappa}{N} \end{aligned} \tag{4.7}$$

On the other hand:

$$\left\langle\left(f_m(x)-\frac{(N-1)\,\kappa}{N}\right)^2\right\rangle=\int_{\Omega^N}\left(f_m(x)-\frac{(N-1)\,\kappa}{N}\right)^2 dx_1\cdots dx_N$$
$$=\frac{(N-1)}{N^2}\left[\int_{\Omega}(K(\xi,0))^2\,d\xi-\left(\int_{\Omega}K(\xi,0)\,d\xi\right)^2\right]$$
$$\equiv\frac{(N-1)\,\Xi}{N^2} \tag{4.8}$$

Let us write:

$$\mathcal{U}_m=\left\{x\in\Omega^N : f_m(x)\leqslant(\theta+1)\,\frac{(N-1)\,\kappa}{N}\right\} \tag{4.9}$$

Using the classical Chebyshev's inequality, we deduce that the probability of the complementary of the set $\mathcal{U}_m$ can be estimated as follows:

$$p((\mathcal{U}_m)^c)=p\left\{x\in\Omega^N : f_m(x)>(\theta+1)\,\frac{(N-1)\,\kappa}{N}\right\}$$
$$\times p\left\{x\in\Omega^N : \left|f_m(x)-\frac{(N-1)\,\kappa}{N}\right|\geqslant\theta\,\frac{(N-1)\,\kappa}{N}\right\}$$
$$\leqslant\frac{\Xi}{\theta^2\kappa^2}\,\frac{1}{(N-1)} \tag{4.10}$$

By (4.10), it follows that $p(\mathcal{U}_m)\geqslant 1-(\Xi/\theta^2\kappa^2)(1/(N-1))$. We are interested in estimating the probability of the set $\bigcap_{m=1}^{N}(\mathcal{U}_m)$, since one has there that $\max_{m=1,\ldots,N}(f_m(x))\leqslant(\theta+1)((N-1)\,\kappa/N)$. Iterating the formula $p(A\cap B)\geqslant p(A)+p(B)-1$, we obtain:

$$p\left(\bigcap_{m=1}^{N}(\mathcal{U}_m)\right)\geqslant 1-\frac{\Xi}{\theta^2\kappa^2}\,\frac{N}{(N-1)} \tag{4.11}$$

whence:

$$\max_{m=1,\ldots,N}\left(\frac{1}{N}\sum_{\ell=1,\,\ell\neq m}^{N}K(x_\ell,x_m)\right)\leqslant C(\theta+1) \tag{4.12}$$

in a set whose probability satisfies (4.2). This inequality allows us to estimate the right hand side of (4.6). In order to conclude our analysis of the terms in (4.4), we just remark that:

$$\sum_{m=1}^{N} \left| \chi(\xi_m(t)) - \frac{N(t)\, \xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)} \right| \leqslant N(t) \tag{4.13}$$

since each term in the sum corresponding to a particle with zero radius gives a zero contribution, and each contribution is bounded by one.

Combining (4.4), (4.6) and (4.13) we obtain:

$$J_1 \leqslant C(\varepsilon_N)^{2/3}\, (N)^{1/3}\, (\theta+1) \sup_{i=1,\dots,N} \xi_i(t) \tag{4.14}$$

in a set satisfying (4.2). We now proceed to estimate J_2. To this end, using (4.12) we obtain:

$$\begin{aligned} &\left(\max_{i=1,\dots,N} \frac{1}{N} \sum_{m=1}^{N} M_{i,m} \left| \chi(\xi_m(t)) - \frac{N(t)\, \xi_m(t)}{\sum_{s=1}^{N} \xi_s(t)} \right| \right) \\ &\quad \leqslant \max_{i=1,\dots,N} \frac{1}{N} \sum_{m=1}^{N} M_{i,m} \leqslant C(\theta+1) \end{aligned} \tag{4.15}$$

that holds in a set satisfying (4.2). Notice that we can assume that (4.14) and (4.15) are satisfied in a common set satisfying (4.2).

On the other hand, we have that:

$$\frac{1}{N(t)} \sum_{\ell=1}^{N} \xi_\ell(t) \leqslant \sup_{i=1,\dots,N} \xi_i(t) \tag{4.16}$$

Putting together (4.5), (4.15) and (4.16) we obtain the estimate:

$$J_2 \leqslant C(\varepsilon_N)^{2/3}\, (N)^{1/3}\, (\theta+1) \sup_{i=1,\dots,N} \xi_i(t) \tag{4.17}$$

whence (4.1) follows by combining (4.14), (4.17), whence Proposition 4.1 follows. ∎

5. STOCHASTIC TERMS DO NOT SERIOUSLY MODIFY SELF-SIMILAR ASYMPTOTICS FOR SMALL VOLUME FRACTIONS

The goal of this section is to show that the stochastic terms that have been computed in previous regions do not modify the long term asymptotics of the distribution of particles, at least during time scales for which the number of particles is large enough as to allow a continuous description of the distribution of radii. Several of the arguments used in this section have many analogies with the ideas introduced in ref. 8, although from a technical point view there are developed in a rather different way that looks better adapted to the analysis of the problem in this paper.

Let us recall the form of the self-similar solutions for classical LSW model. They are solutions with an average radius with the form:

$$\langle \xi \rangle = a(t+1)^{1/3} \tag{5.1}$$

where $a \in (0, (\frac{2}{3})^{2/3}]$. For a in such an interval, the polynomial $P(\eta) = a\eta^3 + 3a - 3\eta$ has two roots in the region $\eta > 0$. Both roots coalesce for $a = (\frac{2}{3})^{2/3}$. Let us denote them as η_1, η_2, $0 < \eta_1 < \eta_2$. The self-similar distribution function for radii satisfying $0 < \xi/(t)^{1/3} \leqslant \eta_1$ is given by:

$$F(\xi, t) = 1 - \frac{1}{(t+1)} G\left(\frac{\xi}{(t+1)^{1/3}}\right), \qquad t \geqslant 0 \tag{5.2}$$

where:

$$G(\eta) = \bar{C} \frac{(\eta_1 - \eta)^{\alpha_1}}{(\eta_2 - \eta)^{\alpha_2}} \left(\eta + \frac{3}{\eta_1 \eta_2}\right)^{-\alpha_3} \tag{5.3}$$

The constant $\bar{C}$ is given by $\bar{C} = [(\eta_2)^{\alpha_2}/(\eta_1)^{\alpha_1}](3/\eta_1\eta_2)^{\alpha_3}$, and the coefficients α_i are given by:

$$\alpha_1 = \frac{3(\eta_1)^3\, \eta_2}{(3 + (\eta_1)^2\, \eta_2)(\eta_2 - \eta_1)} = \frac{a(\eta_1)^2}{(1 - a(\eta_1)^2)} \tag{5.4}$$

$$\alpha_2 = \frac{3(\eta_2)^3\, \eta_1}{(3 + (\eta_2)^2\, \eta_1)(\eta_2 - \eta_1)} \tag{5.5}$$

$$\alpha_3 = \frac{27}{9 + 3(\eta_2)^2\, \eta_1 + 3(\eta_1)^2\, \eta_2 + (\eta_1)^3\, (\eta_2)^3} \tag{5.6}$$

On the other hand $F(\xi, t) = 1$ if $\xi/(t)^{1/3} \geqslant \eta_1$. Notice that $F(0, 0) = 0$. Elementary arguments show that:

$$\frac{\partial \eta_1}{\partial a} > 0, \qquad a \in \left(0, \left(\frac{2}{3}\right)^{2/3}\right) \tag{5.7}$$

and also

$$\lim_{a \to 0^+} \frac{\eta_1}{a} = 1 \tag{5.8}$$

$$\lim_{a \to ((2/3)^{2/3})^-} \eta_1 = \left(\frac{3}{2}\right)^{1/3} \tag{5.9}$$

Notice that (5.7) easily implies:

$$a(\eta_1)^2 < 1 \tag{5.10}$$

Formula (5.2) is valid only of $a \in (0, (\frac{2}{3})^{2/3})$. As a matter of fact, the LSW theory assumes that the physical distribution of particles is the corresponding to $a = (\frac{2}{3})^{2/3}$, that will not be described here.

In ref. 8, it has been shown that for any initial distribution of radii that behaves near the maximum radius as the power law $K(\xi_0 - \xi)^{\alpha_1}$, where α_1 is as in (5.4) the long time asymptotics of the solution of the LSW model as $t \to \infty$ is described by the self-similar solution (5.2). In this section we shall prove that an analogous result can be derived for the model (2.54), (2.55), taking as initial data, instead of a function with a power law asymptotics, a distribution function associated to a finite number of particles as has been described in Section 3. On the other hand, we will not consider the asymptotic behaviour as $t \to \infty$, since for such large times only one particle remains. Instead we consider the asymptotic of solutions for t large, but not so long that less than $\log(N)$ particles are left. After such times, the effect of stochastic fluctuations should become very relevant in the description of the asymptotic of solutions. This last is a most interesting problem, that however, will not be considered in this paper.

We introduce the following smallness assumption for ε_N:

(H) If α_1 in (5.4) is larger or equal that one we require $\varepsilon_N \ll 1/N^2$. If α_1 in (5.4) is less than one, we will assume that $\varepsilon_N \ll 1/N^{(1/2)(1+3/\alpha_1)+\delta}$, where $\delta > 0$ is a fixed, arbitrarily small number.

The main result that will be proved in this section is the following:

Theorem 5.1. Assume that the hypothesis (H) above holds. Let us take a distribution of N spherical particles according to the distribution law (2.21), where $f(\xi, 0)\, d\xi = dF(\xi, 0)$ and $F(\xi, 0)$ is as in (5.2), (5.3). Suppose that the corresponding initial radii are given by the numbers $\{\xi_i(0)\}$.

Denote as $\xi_i(t)$ the solution of (2.54), (2.55) with this initial distribution of radii. Let us define the discrete distribution of particles $F_N(\xi, t)$ as:

$$F_N(\xi, t) = \frac{1}{N} \sum_{i=1}^{N} \chi(\xi - \xi_i(t)) \tag{5.11}$$

Finally, let us fix T_N satisfying:

$$1 \ll T_N \ll \frac{N}{\log(N)} \tag{5.12}$$

Then, with probability one, there holds:

$$\lim_{N \to \infty} \sup_{0 \leqslant t < T_N} [t \, |F_N(\xi, t) - F(\xi, t)|] = 0 \tag{5.13}$$

Theorem 5.1 means that the discrete set of particles can be described (with probability close to one) by the self-similar solutions in (5.2), (5.3) if the number of particles left is larger than $\log(N)$. In particular, this result implies under the assumption (H) that neither the stochastic deviations of the mean field theory that have been included in (2.54), (2.55), or the stochastic fluctuations in the distribution of radii can include any of the self-similar solutions of the LSW theory. Very likely the asymptotics (5.13) holds for long times, if as initial distribution is taken a function $F(\xi, 0)$ with an asymptotics near the maximum radius analogous to the self-similar solutions (5.2), (5.3). The proof of this result would probably use a line of reasoning analogous to the one in ref. 8. However, since the main goal of this paper is just to estimate the effect of stochastic fluctuations in the dynamics of the LSW model, this path will not be pursued here.

We also remark that in the course of the proof of Theorem 5.1, estimates more precise that (5.13) will be derived, in particular near the region of radii close to the maximum one.

The proof of Theorem 5.1 will be made in several steps. Without loss of generality, we can relabel the index of the initial distribution of particles in order to have:

$$\xi_1(0) > \xi_2(0) > \cdots > \xi_N(0) \tag{5.14}$$

Inequality (5.14) will not continue being true in general for positive times due to the terms $\lambda_i(t)$ in (2.54). Notice also that we can assume strict inequalities in (5.14), except for a set of zero probability.

It is convenient to rewrite (2.54) using a new set of variables:

$$\xi_i(t) = \xi_1(t)\, \Phi_i(t) \tag{5.15}$$

$$d\tau = \frac{dt}{(\xi_1(t))^3\, ((1/N(t)) \sum_{j=1}^{N} \Phi_i)} \tag{5.16}$$

where, by convenience we take the normalization $\tau = 0$ at $t = 0$. Then equation (2.54) becomes:

$$\begin{aligned} &\left(\frac{1}{N(t)} \sum_{j=1}^{N} \Phi_i\right) (\xi_1(t))^2 \frac{d\xi_1(t)}{dt} \Phi_i + \frac{d\Phi_i}{d\tau} \\ &\quad = -\frac{((1/N(t)) \sum_{j=1}^{N} \Phi_i)}{(\Phi_i)^2} + \frac{1+\lambda_i(t)}{\Phi_i}, \qquad i = 1,..., N \end{aligned} \tag{5.17}$$

It follows by (5.15) that $\Phi_1 = 1$, whence (5.17) implies:

$$\left(\frac{1}{N(t)} \sum_{j=1}^{N} \Phi_i\right) (\xi_1(t))^2 \frac{d\xi_1(t)}{dt} = -\left(\frac{1}{N(t)} \sum_{j=1}^{N} \Phi_i\right) + (1 + \lambda_1(t)) \tag{5.18}$$

Using (5.18), (5.17) becomes:

$$\frac{d\Phi_i}{d\tau} = \left(\frac{1}{N(t)} \sum_{j=1}^{N} \Phi_i\right)\left(\Phi_i - \frac{1}{(\Phi_i)^2}\right) + \left[\frac{1+\lambda_i(t)}{\Phi_i} - (1 + \lambda_1(t))\, \Phi_i\right] \tag{5.19}$$

where $i = 1,..., N$. The reformulation of the problem (5.19) is only valid as far as $\xi_1(t)$ does not vanish (cf. (5.16)). However, since the ordering in (5.14) is not preserved for $t > 0$, $\xi_1(t)$ could vanish when there are still particles left, due to the (probabilistic) smallness of the terms $\lambda_i(t)$. It will be checked later that the vanishing of $\xi_1(t)$ cannot occur (in a probabilistic sense) before the time T_N.

A crucial estimate in the proof of Theorem 5.1 is the following:

Proposition 5.2. Let us consider the solution of the differential equation:

$$\Phi_\tau = v(\tau) \left(\Phi - \frac{1}{\Phi^2}\right) + \left(\frac{1}{\Phi} - \Phi\right) \tag{5.20}$$

with initial data:

$$\Phi(0) = \beta > 0 \tag{5.21}$$

where, in (5.20) $v(\tau)$, is a function bounded in each time interval. Then, as far as $\Phi(\tau)$ remains positive, the following identity holds:

$$\frac{\partial\Phi}{\partial\beta}=\frac{\beta^2(1-\Phi^3)}{\Phi^2(1-\beta^3)}\exp\left(\int_0^\tau W(\Phi(\sigma))\,d\sigma\right) \tag{5.22}$$

where $W(r)=(1+r-2r^2)/((r^2+r+1)\,r^2)$. The function $W(r)$ is singular at $r=0$, decreases in the interval $(0, 1)$ and vanishes at $r=1$.

Proof of Proposition 5.2. Let us define Z as follows:

$$Z=\frac{\partial\Phi}{\partial\beta}$$

On differentiating (5.20), it is easily checked, that Z solves:

$$\frac{dZ}{d\tau}=v(\tau)\left(1+\frac{2}{\Phi^3}\right)Z-\left(1+\frac{1}{\Phi^2}\right)Z \tag{5.23}$$

$$Z(0)=1 \tag{5.24}$$

Eliminating $v(\tau)$ in (5.23) with the help of (5.20), we obtain:

$$\frac{1}{Z}\frac{dZ}{d\tau}=\left(-\frac{2}{\Phi}+\frac{3\Phi^2}{(\Phi^3-1)}\right)\frac{d\Phi}{d\tau}-W(\Phi) \tag{5.25}$$

Integrating (5.25) with the initial conditions (5.21), (5.24) we deduce (5.22). Then, the required properties of function $W(r)$ can be easily proved by means of standard computations. ∎

Proposition 5.2 might be used to analyse the asymptotics of solutions of the LSW model. Since we are interested in the modified LSW model (5.19), we need a more refined version of Proposition 5.2 that we formulate as follows:

Proposition 5.3. Let us consider two functions $\Phi_1=\Phi(\tau,\beta_1)$, $\Phi_1=\Phi(\tau,\beta_2)$ that solve respectively the differential equations:

$$\frac{d\Phi_i}{d\tau}=v(\tau)\left(\Phi_i-\frac{1}{(\Phi_i)^2}\right)+\left[\frac{1+\mu_\alpha(\tau;\beta_i)}{\Phi_i}-(1+\mu_\beta(\tau;\beta_i))\,\Phi_i\right] \tag{5.26}$$

$$\Phi_i(0)=\beta_i \tag{5.27}$$

where $\beta_i \in (0, 1)$, $i=1, 2$ and the functions $v(\tau)$, $\mu_\alpha(t; \beta_i)$, $\mu_\beta(t; \beta_i)$ are bounded in each bounded interval. Then, as far as both functions $\Phi(\tau, \beta_i)$ are different from zero, there holds:

$$\Psi(\tau) = \frac{(\beta_2 - \beta_1)(\beta_1)^2\,(1-(\Phi_1(\tau))^3)}{(\Phi_1(\tau))^2\,(1-(\beta_1)^3)} \exp\left(\int_0^\tau Y(\sigma)\,d\sigma\right) + \int_0^\tau \frac{(\Phi_1(s))^2\,(1-(\Phi_1(\tau))^3)}{(\Phi_1(\tau))^2\,(1-(\Phi_1(s))^3)}\,d(s)\exp\left(\int_s^\tau Y(\sigma)\,d\sigma\right) ds \tag{5.28}$$

where $W(r)$ is as in 5.2 and:

$$\Psi(\tau) = \Phi_2(\tau) - \Phi_1(\tau) \tag{5.29}$$

$$Y(\sigma) = [\,W(\Phi_1(\sigma)) + b(\sigma) + c(\sigma)\,\Psi(\sigma)] \tag{5.30}$$

$$b(\tau) = \frac{(2+(\Phi_1)^3)}{\Phi_1((\Phi_1)^3 - 1)}\left[\frac{\mu_\alpha(t;\beta_1)}{\Phi_1} - \mu_\beta(t;\beta_1)\,\Phi_1\right] - \left[\frac{\mu_\alpha(t;\beta_1)}{\Phi_1\Phi_2} + \mu_\beta(t;\beta_1)\right] \tag{5.31}$$

$$c(\tau) = -\frac{2v(\tau)}{(\Phi_1)^3\,(\Phi_2)^2}(\Phi_1+\Phi_2) + \frac{v(\tau)}{(\Phi_1)^2\,(\Phi_2)^2} + \frac{1}{(\Phi_1)^2\,\Phi_2} \tag{5.32}$$

$$d(\tau) = \left[\frac{\mu_\alpha(t;\beta_2) - \mu_\alpha(t;\beta_1)}{\Phi_2} - (\mu_\beta(t;\beta_2) - \mu_\beta(t;\beta_1))\,\Phi_2\right] \tag{5.33}$$

Estimate (5.28) is just a generalization of (5.22) to the case when the effect of the corrective terms in (5.26) that have not been included in (5.22) is taken into account.

Proof of Proposition 5.3. The proof of this result is just an adaptation of that of Proposition 5.2. We just keep track of the additional terms that are in (5.26) and were not in (5.20). Using (5.26) and (5.29), we readily obtain:

$$\frac{d\Psi}{d\tau} = v(\tau)\left(1 + \frac{\Phi_1+\Phi_2}{(\Phi_1\Phi_2)^2}\right)\Psi - \left(\frac{1+\mu_\alpha(\tau;\beta_1)}{\Phi_1\Phi_2} + (1+\mu_\beta(\tau;\beta_1))\right)\Psi + \left(\frac{\mu_\alpha(t;\beta_2) - \mu_\alpha(t;\beta_1)}{\Phi_2} - (\mu_\beta(t;\beta_2) - \mu_\beta(t;\beta_1))\,\Phi_2\right) \tag{5.34}$$

Eliminating $v(\tau)$ from the first term on the right hand side of (5.34), we then arrive, after some simple computations, at:

$$\frac{d\Psi}{d\tau} = \frac{2 + (\Phi_1)^3}{\Phi_1((\Phi_1)^3 - 1)} \frac{d\Phi_1}{d\tau} \Psi - W(\Phi_1)\,\Psi + b(\tau)\,\Psi + c(\tau)(\Psi)^2 + d(\tau) \tag{5.35}$$

$$\Psi(0) = \beta_2 - \beta_1 \tag{5.36}$$

where we have used also (5.27). We then easily deduce (5.28) integrating the ODE (5.35), (5.36) by means of the standard variation of constants formula for linear equations. ▮

To proceed further with the proof of Theorem 5.1, we decompose the space of radii as follows. Assume that function $1 - F(\xi, 0)$ is supported in the interval $[0, \bar{\xi}_0]$. Let us denote as ρ_N the number $N/[T_N]$, where T_N is as in the statement of Theorem 5.1, and $[\,\cdot\,]$ is the integer part of its argument. Notice that (5.12) implies:

$$\log(N) \ll \rho_N \sim \frac{N}{T_N} \ll N \tag{5.37}$$

We now pick a sequence of integers J_N satisfying:

$$1 \ll J_N \ll \frac{\rho_N}{\log(N)} \tag{5.38}$$

We then define:

$$R_N = \frac{\rho_N}{J_N} \tag{5.39}$$

Taking into account (5.37), (5.38), it readily follows that:

$$\log(N) \ll R_N \ll \rho_N \ll N \tag{5.40}$$

Moreover, notice that $M_N = N/R_N = J_N[T_N]$ is an integer.

We now split the interval $[0, \bar{\xi}_0]$ as in (3.11), (3.12). Putting together J_N of the subintervals in (3.12), we obtain a coarser decomposition of $[0, \bar{\xi}_0]$. We then obtain the following:

$$[0, \bar{\xi}_0] = \bigcup_{\ell=1}^{[T_N]} [\zeta_\ell, \zeta_{\ell-1}] \tag{5.41}$$

where:

$$\zeta_\ell = \xi_{J_N \ell}, \qquad \ell = 0, \ldots, [T_N] \tag{5.42}$$

It is immediately seen that:

$$[\zeta_\ell, \zeta_{\ell-1}] = \bigcup_{j=1}^{J_N} [\xi_{J_N(\ell-1)+j}, \xi_{J_N(\ell-1)+j-1}] \tag{5.43}$$

In other words, each interval $[\zeta_\ell, \zeta_{\ell-1}]$ has been divided in J_N subintervals. Taking into account (3.11) and (5.39), we readily deduce:

$$1 - F(\zeta_\ell) = \frac{\rho_N}{N} \ell \tag{5.44}$$

As a next step, it is convenient to describe the self-similar solutions given in (5.2), (5.3) using the variables Φ_i, τ, that have been introduced in (5.15), (5.16). Notice that by (5.1), (5.15) the average value of Φ would be:

$$\langle \Phi \rangle = \frac{a}{\eta_1} \tag{5.45}$$

Standard computations that use also the equation $P(\eta_1) = 0$ show that:

$$\frac{d}{da}(\langle \Phi \rangle) = \frac{9(a - \eta_1)}{3(\eta_1)^2 (1 - a(\eta_1)^2)} \tag{5.46}$$

On the other hand, since $P(a) = a^4 > 0$, it readily follows that $\eta_1 > a$, whence, taking also into account (5.7), we obtain:

$$\frac{d}{da}(\langle \Phi \rangle) < 0 \tag{5.47}$$

From (5.8), (5.9) and (5.45) we derive the asymptotics:

$$\lim_{a \to 0^+} \langle \Phi \rangle = 1 \tag{5.48}$$

$$\lim_{a \to ((2/3)^{2/3})^-} \langle \Phi \rangle = \tfrac{2}{3} \tag{5.49}$$

Let us denote as $v(\tau)$ the average value $((1/N(t)) \sum_{j=1}^{N} \Phi_j)$ in (5.19). For one of the self-similar solutions given in (5.2), (5.3), the function $v(\tau)$

takes the constant value given in (5.45), and that we will denote as v_0. By (5.48), (5.49), we have that $v_0 \in (2/3, 1)$.

It will be useful to notice that in the particular case where $v(\tau) = v_0 \in (2/3, 1)$, an explicit solution of (5.20), (5.21) can be written. Indeed, let us suppose that we define a constant χ by means of the formula:

$$\chi = G(\eta_1 \beta) \tag{5.50}$$

where $G(\cdot)$ is as in (5.3) and β is as in (5.21). Then, the solution of (5.20), (5.21) is given by the formula:

$$\chi(t+1) = G(\eta_1 \Phi(\tau)) \tag{5.51}$$

where t and τ are related by means of (5.16). In order to check this, it is enough to notice that χ is a constant of integration for the differential equation (5.20). In fact, using (5.16) and (5.20) it can be readily seen that:

$$\chi_\tau = \frac{v_0(\eta_1)^2}{(t+1)} \left[-G + \left(\frac{(v_0 - 1)}{v_0(\eta_1)^3} (\eta_1 \Phi(\tau)) + \frac{1}{v_0 \eta_1} \frac{1}{\eta_1 \Phi(\tau)} - \frac{1}{(\eta_1 \Phi(\tau))^2} \right) G' \right]$$

Using (5.3), it follows that $\chi_\tau = 0$, whence the desired claim follows.

We will assume that $v(\tau)$ remains close to v_0 during its whole evolution (and always in the interval $(\frac{2}{3}, 1)$). Certainly, this can be made if the distribution of radii remains close enough to one of the self-similar solutions given in (5.2), (5.3). As a matter of fact we will show that $v(\tau)$ remains close to v_0 in this way with a standard bootstrap argument. More precisely we will assume that $v(\tau) \in (v_0 - \varepsilon_0, v_0 + \varepsilon_0)$ for some $\varepsilon_0 > 0$, and that the initial distribution of radii is very close to the self-similar solution of the family (5.2), (5.3) with the corresponding average v_0. It will be shown that under these assumptions the radii Φ_i evolve in such a way that the self-similar distribution is approximately kept as well as the inequalities $v_0 - \varepsilon_0 < v(\tau) + \varepsilon_0$ during the forthcoming evolution. In a similar way we will prove the estimate:

$$\xi_1(t) \leqslant C(t)^{1/3} \tag{5.52}$$

for some suitable constant $C > 0$ independent on N. We will assume (5.52) and we will prove that this bound is recovered with a strictly smaller bound C for later times, obtaining in this form a proof of (5.52).

It is convenient to define the evolution of the intervals $[\bar{\xi}_\ell, \bar{\xi}_{\ell-1}]$, $\ell = 1, \ldots, M_N$, that have been defined in (3.11), (3.12). To this end, we just make evolve each value ξ by means of the ODE (2.54), where as function $\lambda_i(t)$ there we just take the value of $\lambda_i(t)$ at the closest particle $\xi_i(t)$ to the right of the value $\xi(t)$. Here we denote as $\xi(t)$ the value of the solution of (2.54) at time t with initial data $\underline{\xi}$ and $\lambda_i(t)$ as indicated. As we have defined the evolution of each point $\bar{\xi}_j$ we can then define the evolution of the whole intervals $[\bar{\xi}_\ell, \bar{\xi}_{\ell-1}]$ by means of the evolution of their extreme points. Notice that in principle the points $\bar{\xi}_\ell$, $\bar{\xi}_{\ell-1}$ could reverse their ordering. The evolution of intervals in the Φ variable will be made in an analogous way. For convenience, let us write:

$$\bar{\Phi}_\ell = \frac{\bar{\xi}_\ell}{\eta_1} \tag{5.53}$$

and let us denote as $T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ the evolution of the interval $[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ by means of the previous law.

Our next goal is to show that the evolution of the intervals $T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ preserve to some degree some of the most relevant properties of the evolution of intervals under the ODE (5.20). More precisely, disjoint intervals remain disjoint under the evolution given by (5.20). Although this cannot be asserted for the evolution of intervals of the form $[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ described above, it turns out that the interaction between them is rather small. Also, a discrete version of (5.22) holds. As a preliminary step we prove a rough estimate for the size of the intervals $T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$:

Lemma 5.4. Let us assume that for all $\tau \geqslant \tau_0$, $v(\tau) = (1/N(t)) \sum_{j=1}^{N} \Phi_j \in (v_0 - \varepsilon_0, v_0 + \varepsilon_0)$, where $\frac{2}{3} < v_0 - \varepsilon_0 < v_0 + \varepsilon_0 < 1$, and ε_0 is small enough (depending on δ in assumption (H)). Suppose also that the hypothesis (H) and (5.52) are satisfied, and also that $0 \leqslant t \leqslant N$. Then, there exists an integer ℓ_0 independent on N such that, with a probability that can be made arbitrarily close to one if N is large, we have:

$$\begin{aligned} \operatorname{diam}([\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}])\, e^{[3v_0 - 2 - C\varepsilon_0]\,\tau} &\leqslant \operatorname{diam}(T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]) \\ &\leqslant \operatorname{diam}([\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}])\, e^{[3v_0 - 2 + C\varepsilon_0]\,\tau} \end{aligned} \tag{5.54}$$

during the range of times for which the interval $T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ is contained in the set $\Phi \in [\frac{1}{2}, 1]$.

Proof. As a preliminary step we derive an estimate for $\xi_1(t)$. Integrating the ODE (5.18) and taking into account (4.1) we immediately obtain:

$$\left| (\xi_1(t))^3 - (\xi_1(0))^3 - \frac{3(1 - v_0)}{v_0}\, t \right| \leqslant 6\varepsilon_0 t \tag{5.55}$$

where we assume that N is large enough, and (5.55) holds with a probability close to one. From now on, we will make these assumptions without explicitly stating them.

Using (5.16) as well as Proposition 3.1 (in order to estimate $\xi_1(0)$) we deduce with probability close to one that:

$$\frac{1}{C}\, e^{[3(1-v_0)-C\varepsilon_0]\,\tau} \leqslant t \leqslant C e^{[3(1-v_0)+C\varepsilon_0]\,\tau} \tag{5.56}$$

where from now on $C>0$ is a constant independent on N, t, that can possibly change from line to line.

Let us assume that the evolution of $\bar{\Phi}_\ell$, $\bar{\Phi}_{\ell-1}$ is denoted by the functions $\Phi_1(t)$, $\Phi_2(t)$ in Proposition 5.3 respectively. Notice that functions $b(\tau)$, $c(\tau)$, $d(\tau)$ in (5.31)–(5.33) can be estimated under our current assumptions as follows:

$$|b(\tau)| \leqslant C(\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta(t+1)^{1/3} \tag{5.57}$$

$$|c(\tau)| \leqslant C \tag{5.58}$$

$$|d(\tau)| \leqslant C(\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta(t+1)^{1/3} \tag{5.59}$$

where θ (that appears in (4.1)) has been chosen large enough. Taking into account our choice of $\bar{\xi}_\ell$ in (3.11), (3.12), and also the asymptotics of $G(\eta)$ (cf. (5.3)), it follows that:

$$|\bar{\xi}_\ell - \bar{\xi}_{\ell-1}| \leqslant C\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1 - 1} \frac{R_N}{N} \tag{5.60}$$

We now derive an estimate for the time that takes for any of the functions Φ_1, Φ_2 solving (5.26) to become of order one. Let us write $\Phi_i = 1+\varphi$. Linearising in (5.26) and using (4.1), we arrive at:

$$\frac{d\varphi}{d\tau} = 3v(\tau)\,\varphi - 2\varphi + O((\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta(t+1)^{1/3}) \tag{5.61}$$

Integrating (5.61) with the help of (5.56) and recalling the assumptions in 5.4, we obtain:

$$\begin{aligned}(|1-\Phi_i|\, e^{[3v_0-2-C\varepsilon_0]\,\tau} - C(\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta e^{\beta-\tau})&\\ \leqslant |\varphi| \leqslant (|1-\Phi_i|\, e^{[3v_0-2+C\varepsilon_0]\,\tau} + C(\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta e^{\beta+\tau})&\end{aligned} \tag{5.62}$$

where $\beta_{-}=\max\{3v_0-2,\, 1-v_0\}-C\varepsilon_0$, $\beta_{+}=\max\{3v_0-2,\, 1-v_0\}+C\varepsilon_0$. Taking into account the choice of initial data for the functions Φ_i it follows that:

$$C\left(\frac{(\ell-1)\,R_N}{N}\right)^{1/\alpha_1}\leqslant|1-\Phi_i|\leqslant C\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1} \tag{5.63}$$

Combining (5.62), (5.63) and using the assumption (H), we immediately deduce that the terms of order $(\varepsilon_N)^{2/3}\,(N)^{1/3}$ are negligible if $\ell\geqslant 2$. In particular this implies that the difference Φ_i-1 becomes of order one if:

$$\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1} e^{[3v_0-2-C\varepsilon_0]\,\tau}\approx 1 \tag{5.64}$$

We now use a classical continuation argument. It is readily seen that Ψ defined in (5.29) satisfies at $t=0$ the estimate (cf. (5.60)):

$$|\Psi|\leqslant L\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1-1}\frac{R_N}{N}\,e^{[3v_0-2+C\varepsilon_0]\,\tau} \tag{5.65}$$

where L is a large constant independent on N, τ to be precised later.

As far as the estimate (5.65) is satisfied we can obtain from (5.28) and (5.57)–(5.59) the bound:

$$\begin{aligned}|\Psi|\leqslant{}& C\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1-1}\frac{R_N}{N}\,\frac{(1-(\bar{\Phi}_1)^3)}{(1-(\Phi_1(\tau))^3)}\\ &\times\exp\left(\int_0^\tau W(\Phi_1(\sigma))\,d\sigma+\frac{CL}{\ell}\right)+C\int_0^\tau\frac{(1-(\Phi_1(s))^3)}{(1-(\Phi_1(\tau))^3)}\,|d(s)|\\ &\times\exp\left(\int_s^\tau W(\Phi_1(\sigma))\,d\sigma+\frac{CL}{\ell}\right)ds\end{aligned} \tag{5.66}$$

where the constant C is independent on L. Taking into account the definition of $W(\Phi)$, it can be easily checked that the integral term $\int_0^\tau W(\Phi_1(\sigma))\,d\sigma$ is uniformly bounded. Then, using (5.59), it is not hard to derive the estimate:

$$|\Psi|\leqslant C\left[\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1-1}\frac{R_N}{N}\,e^{[3v_0-2+C\varepsilon_0]\,\tau}+(\varepsilon_N)^{2/3}\,(N)^{1/3}\,\theta(t+1)^{1/3}\right] \tag{5.67}$$

where $\ell \geqslant \ell_0$ and ℓ_0 is an integer dependent on L but not on θ, N, t. Using the assumption (H) it follows that for $t \leqslant N$, and N large enough:

$$|\Psi| \leqslant C\left(\frac{\ell R_N}{N}\right)^{1/\alpha_1 - 1} \frac{R_N}{N}\, e^{[3\nu_0 - 2 + C\varepsilon_0]\,\tau} \tag{5.68}$$

Since in (5.68) we recover (5.65) with a new constant C independent on L, a standard continuation argument shows that this estimate is valid not only at $t=0$, but for any t in the interval $[0, N]$. This finishes the proof of the upper estimate for the diameter of $T_\tau[\bar{\Phi}_\ell, \bar{\Phi}_{\ell-1}]$ in (5.54). The proof of the lower estimate can be made exactly along similar lines, whence the proof of Lemma 5.4 is concluded. ∎

We also have the following result:

Lemma 5.5. Under our current assumptions, suppose that one particle has an initial radius $\xi_j(0)$ in the interval $(\bar{\xi}_\ell, \bar{\xi}_{\ell-1})$, $\ell = 1,..., M_N$. Then the subsequent radius $\xi_j(t)$ belongs to the set $T_\tau(\bar{\xi}_{\ell+1}, \bar{\xi}_{\ell-2})$, where by definition $\bar{\xi}_{-1} = \bar{\xi}_0$. Moreover, the subsequent points $T_\tau(\bar{\xi}_\ell)$ remain ordered during their whole evolution, i.e.:

$$T_\tau(\bar{\xi}_\ell) < T_\tau(\bar{\xi}_{\ell-1}), \qquad \ell = 1,..., M_N \tag{5.69}$$

Proof. The proof of Lemma 5.5 is rather similar to that of Lemma 5.4. Indeed, let us remark that writing the evolution equations for the functions $\Phi_1 = \xi_j(t)/\xi_1(t)$ and $\Phi_2 = \bar{\xi}_k(t)/\xi_1(t)$ where k takes the values $\ell-1$, ℓ we can derive an estimate similar to (5.66). As in Lemma 5.4, the second term in the right-hand side there is negligible compared with the first one under the assumption (H). The first term in the right-hand side of (5.66) is exactly of the order of magnitude of any of the intervals $T_\tau(\bar{\xi}_\ell, \bar{\xi}_{\ell-1})$, $T_\tau(\bar{\xi}_{\ell-1}, \bar{\xi}_{\ell-2})$, $T_\tau(\bar{\xi}_{\ell+1}, \bar{\xi}_\ell)$. It is then not hard to check that the particle $\xi_j(t)$ remains inside $T_\tau(\bar{\xi}_{\ell+1}, \bar{\xi}_{\ell-2})$ during the whole range of times considered in Lemma 5.4. The proof of (5.69) is just a consequence of the fact that the last term in (5.66) is negligible when compared with the previous one, and then it cannot alter the relative order of these points. ∎

We can now consider the evolution of the whole group of intervals $[\zeta_\ell, \zeta_{\ell-1}]$. By means of Lemmata 5.4, 5.5 we can show that the diffusion of particles through the boundaries of these sets is a negligible effect. For convenience we recall that the intervals $[\zeta_\ell, \zeta_{\ell-1}]$ have been decomposed in a set of intervals (cf. 5.43). We define intervals $[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}] \subset [\zeta_\ell, \zeta_{\ell-1}]$ as:

$$[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}] = \bigcup_{j=2}^{J_N-1} [\xi_{J_N(\ell-1)+j}, \xi_{J_N(\ell-1)+j-1}] \tag{5.70}$$

We can now state the main features of the evolution of the intervals $[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$.

Lemma 5.6. Under our current assumptions, the intervals $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ remain disjoint during their whole evolution for times $0 \leqslant t \leqslant N$, with probability close to one as $N \to \infty$.

Let us denote as $L_\ell(\tau)$ the length of each interval $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$. If we denote as $\Phi_\ell(\tau)$ the evolution of the point $\hat{\zeta}_\ell$, then the following identity holds:

$$L_\ell(\tau) = \frac{L_\ell(0)(\hat{\Phi}_\ell)^2\,(1-(\Phi_\ell(\tau))^3)}{(\Phi_\ell(\tau))^2\,(1-(\hat{\Phi}_\ell)^3)} \exp\left(\int_0^\tau W(\Phi_\ell(\sigma))\,d\sigma\right)(1+v_{N,\ell}(t)) \tag{5.71}$$

where $v_{N,\ell}(t)$ is a number that can be made arbitrarily small if $N \to \infty$ uniformly in ℓ, t, with probability arbitrarily close to one. The number of particles contained in each interval $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ is $\rho_N(1+\delta_{N,\ell}(t))$, where ρ_N has been defined before and $\delta_{N,\ell}(t)$ can be made uniformly small with probability close to one if $N \gg 1$.

Proof. The fact that the intervals $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ remain disjoint during their evolution is just a consequence of Lemma 5.5. Indeed, at $\tau = 0$ the intervals $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ are separated between themselves by subintervals of the form $[\xi_{J_N(\ell-1)+j}, \xi_{J_N(\ell-1)+j-1}]$ with $j = 1, J_N$. It then follows that the intervals $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ are contained in the intervals $[T_\tau(\zeta_\ell), T_\tau(\zeta_{\ell-1})]$, whence the disjointness of the sets $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ follows. The proof of (5.71) is completely similar to the one of Lemma 5.4, taking into account the smallness of the last term in (5.66) under the assumption (H). Notice to this end that by (5.38) the number ℓ involved is larger that ℓ_0 as $N \to \infty$. Finally the statement about the number of particles is a consequence of (3.15), (5.40). ∎

We now need to analyse the effect that the evolution of the particles produces in the average value $v(\tau) = (1/N(t)) \sum_{j=1}^N \Phi_j(t)$. For $t = 0$, $v(\tau)$ is arbitrarily close to v_0 as $N \to \infty$, with probability one, due to the classical large numbers law. In order to estimate $v(\tau)$ we will restrict our attention to the particles included in the intervals $T_\tau(\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$, because those remaining in the sets $T_\tau[\zeta_\ell, \zeta_{\ell-1}] \backslash T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ are a negligible number. In fact, due to Proposition 3.1 and the choice of ρ_N, R_N (cf. (5.37)–(5.39)), as well as Lemma 5.6, it follows that the number of particles in these last sets is of order:

$$2\,\frac{N(t)}{\rho_N}\,R_N \ll N(t) \tag{5.72}$$

where, as usual $N(t)$ is the number of remaining particles. It then follows that during the desired range of times the following approximation holds:

$$v(\tau) = \frac{(1+o(1))}{N(t)} \sum_{\{\Phi_j(t) \in \bigcup_\ell T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]\}} \Phi_j(t) \tag{5.73}$$

as $N \to \infty$ with probability arbitrarily close to one.

Before concluding the proof of Theorem 5.1, it is instructive to describe the main argument in the continuous case (without fluctuations) since it is simpler. We will work locally in a neighbourhood of $v(\tau) \approx v_0$. Let us write:

$$v(\tau) = v_0 + \lambda(\tau) \tag{5.74}$$

Keeping in (5.20) the linear terms in $\lambda(\tau)$ only, we would obtain after some simple manipulations:

$$\begin{aligned}\int_0^\tau W(\Phi(\sigma))\,d\sigma &= \int_\beta^{\Phi(\tau)} \frac{W(\Phi)\,d\Phi}{(v_0(\Phi - 1/\Phi^2) + (1/\Phi - \Phi))} \\ &\quad - \int_0^\tau \frac{W(\Phi(\sigma))(\Phi - 1/\Phi^2)\,\lambda(\sigma)\,d\sigma}{(v_0(\Phi - 1/\Phi^2) + (1/\Phi - \Phi))}\end{aligned} \tag{5.75}$$

Let us denote as $\bar{\Phi}$ the solution of:

$$\frac{d\Phi}{d\sigma} = v_0\left(\Phi - \frac{1}{\Phi^2}\right) + \left(\frac{1}{\Phi} - \Phi\right) \tag{5.76}$$

such that:

$$\bar{\Phi}(\tau) = \Phi(\tau) \tag{5.77}$$

We will write $\bar{\beta} = \bar{\Phi}(0)$. Due to the term $\lambda(\tau)$, $\bar{\beta}$ is in general different from β. Recalling (5.50), (5.51) it follows that:

$$(t+1)\,G(\eta_1 \beta) = G(\eta_1 \bar{\Phi}(\tau)) \tag{5.78}$$

Differentiating (5.78) with respect to β, and using (5.22), (5.77) as well as the differential equation for G we obtain:

$$\exp\left(\int_0^\tau W(\bar{\Phi}(\sigma))\,d\sigma\right) = \exp\left(\int_{\bar{\beta}}^{\bar{\Phi}(\tau)} \frac{W(\Phi)\,d\Phi}{(v_0(\Phi-(1/\Phi^2))+((1/\Phi)-\Phi))}\right)$$
$$= \frac{(1-(\bar{\beta})^3)}{(1-(\bar{\Phi}(\tau))^3)}\,\frac{P(\eta_1\bar{\Phi}(\tau))}{P(\eta_1\bar{\beta})} \tag{5.79}$$

Plugging (5.75), (5.77), (5.79) into (5.22) we arrive at:

$$\frac{\partial\Phi}{\partial\beta} = \frac{\beta^2(1-(\bar{\beta})^3)}{\Phi^2(1-(\beta)^3)}\,\frac{P(\eta_1\Phi)}{P(\eta_1\bar{\beta})}\exp\left(-\int_0^\tau F(\Phi(\sigma))\,\lambda(\sigma)\,d\sigma\right) \tag{5.80}$$

where:

$$F(\Phi) \equiv \frac{W(\Phi)(\Phi-1/\Phi^2)}{(v_0(\Phi-1/\Phi^2)+(1/\Phi-\Phi))} \tag{5.81}$$

In (5.80) we can make two approximations. First, notice that due to (5.77) and the smallness of $\lambda(\tau)$, Φ and $\bar{\Phi}$ are rather close if Φ is at a distance of order one from $\Phi=1$. If $\Phi-1$ is small, the contribution of $F(\Phi)$ will be negligible. This suggests at once the approximation $F(\Phi(\sigma)) \approx F(\bar{\Phi}(\sigma))$. On the other hand, linearizing the terms in $\lambda(\sigma)$ we arrive at:

$$\frac{\partial\Phi}{\partial\beta} = \frac{\beta^2(1-(\bar{\beta})^3)}{\Phi^2(1-(\beta)^3)}\,\frac{P(\eta_1\Phi)}{P(\eta_1\bar{\beta})}\left[1-\int_0^\tau F(\bar{\Phi}(\sigma))\,\lambda(\sigma)\,d\sigma\right] \tag{5.82}$$

From (5.16) and (5.78) it now follows that:

$$G(\eta_1\bar{\Phi}(\sigma)) = G(\eta_1\Phi)\,e^{-(\eta_1)^3 v_0(\tau-\sigma)} \tag{5.83}$$

whence, $F(\bar{\Phi}(\sigma)) = K(\Phi, \tau-\sigma)$.

Now notice that as $\tau \gg 1$, $\bar{\beta}$ approaches one. Taking into account (5.71) we can derive estimates of $v(\tau)$. In view of the previous approximations we can rewrite (5.82) to the leading order as:

$$\frac{3\Phi^2\,d\Phi}{P(\eta_1\Phi)}\left[1+\int_0^\tau K(\Phi,\tau-\sigma)\,\lambda(\sigma)\,d\sigma\right] = \frac{d\beta}{\eta_1(1-a(\eta_1)^2)(1-\beta)} \tag{5.84}$$

For large times only particles with β close to one survive. Then, the initial distribution of particles for radii close to $\beta=1$ is essential in determining the long term distribution of radii (cf. ref. 8). If such initial distribution has the form $G_0(\xi) \approx C(\eta_1 - \eta_1 \beta)^{\alpha_1}$ as $\xi \to \eta_1$, and if we denote as ζ the fraction of radii left above some particular number Φ, it then follows that:

$$\frac{2a\xi^2\, d\xi}{P(\xi)} \left[1 + \int_0^\tau K\left(\frac{\xi}{\eta_1}, \tau - \sigma \right) \lambda(\sigma)\, d\sigma \right] = -\frac{d\zeta}{\zeta} \tag{5.85}$$

where we have introduced the change of variables $\Phi = \xi/\eta_1$

Taking into account that $dG(\xi)/G(\xi) = -(3a\xi^2\, d\xi)/P(\xi)$, where $G(\xi)$ is as in (5.3), we can rewrite (5.85) as:

$$\frac{dG(\xi)}{G(\xi)} + \int_0^\tau \lambda(\sigma)\, d_\xi Z(\xi, \tau - \sigma)\, d\sigma = \frac{d\zeta}{\zeta}$$

where $dZ(\xi, \tau - \sigma) - K(\xi/\eta_1, \tau - \sigma)\, (dG(\xi)/G(\xi))$, and we are using the normalization $Z(0, \tau - \sigma) = 0$. Integrating this last equation, we obtain an expression for the distribution of radii:

$$\zeta(\xi, \tau) = G(\xi) \exp\left(\int_0^\tau Z(\xi, \tau - \sigma)\, \lambda(\sigma)\, d\sigma \right) \tag{5.86}$$

Finally, we remark that $v(\tau) = v_0 + \lambda(\tau) = \int_0^{\eta_1} \zeta(\xi, \tau)\, d\xi$. After a new linearization in $\lambda(\sigma)$, we would deduce the following Volterra integral equation for $\lambda(\tau)$:

$$\lambda(\tau) = \int_0^\tau m(\tau - \sigma)\, \lambda(\sigma)\, d\sigma \tag{5.87}$$

with:

$$m(\tau) \equiv \int_0^{\eta_1} Z(\xi, \tau)\, G(\xi)\, d\xi \tag{5.88}$$

It is not hard to check from the previous computations that $m(\tau)$ is negative and decreasing. In Appendix A at the end of the paper it will be shown that, under these assumptions, the solutions of (5.87) decay exponentially in time. Strictly speaking, the only solution of (5.87) is $\lambda(\tau) \equiv 0$. However it should be taken into account that there are higher order terms that have been neglected in (5.87), and that this equation is only an approximation for long times. Due to this exponential decay, we obtain a fast approximation of the solutions of the LSW model to the self-similar behaviour given in (5.2), (5.3).

End of the Proof of Theorem 5.1. In order to conclude the proof of the theorem, it only remains to extend the argument above to the problem with fluctuations. Basically, we have to retrace the steps in the previous analysis, to adapting them in a suitable way to the system (5.19). Let us denote by γ_N a small positive number that will be precised in the course of the argument, and that essentially provides an upper bound of $\lambda(\tau) = v(\tau) - v_0$, where we understand that $\gamma_N \to 0$ as $N \to \infty$. For each $\Phi_\ell(\tau)$, we define $\bar{\Phi}_\ell(\sigma)$ and $\bar{\beta}_\ell$ exactly as $\bar{\Phi}(\sigma)$ and $\bar{\beta}$ were defined above. As far as $|\lambda(\tau)| \leqslant \gamma_N$, we have the following version of (5.75):

$$\begin{aligned}\int_0^\tau W(\Phi(\sigma))\,d\sigma = \int_\beta^{\Phi(\tau)} \frac{W(\Phi)\,d\Phi}{(v_0(\Phi - 1/\Phi^2) + (1/\Phi - \Phi))} \\ - \int_0^\tau \frac{W(\Phi(\sigma))(\Phi - 1/\Phi^2)\,\lambda(\sigma)\,d\sigma}{(v_0(\Phi - 1/\Phi^2) + (1/\Phi - \Phi))} \\ + \int_0^\tau O\left(\frac{(1 - \Phi(\sigma))(\gamma_N)^2}{(\Phi(\sigma))^2}\right) d\sigma \qquad (5.89)\end{aligned}$$

The last term in (5.89) can be estimated uniformly as $O((\gamma_N)^2)$ with the help of (5.62) as far as $\Phi(\tau) > 0$. Arguing then exactly as in the continuous case, we obtain the following version of (5.82):

$$\begin{aligned}L_\ell(\tau) = \frac{L_\ell(0)(\hat{\Phi}_\ell)^2\,(1 - (\bar{\beta}_\ell)^3)}{(\Phi_\ell(\tau))^2\,(1 - (\hat{\Phi}_\ell)^3)} \frac{P(\eta_1 \Phi_\ell(\tau))}{P(\eta_1 \bar{\beta}_\ell)} \\ \times \exp\left(- \int_0^\tau F(\Phi_\ell(\sigma))\,\lambda(\sigma)\,d\sigma \right)(1 + O((\gamma_N)^2)) \qquad (5.90)\end{aligned}$$

where we have used (5.71), and by assumption the term $(\gamma_N)^2$ dominates the terms $v_{N,\ell}$ there. Using now classical continuous dependence results for ODEs we can approximate $\Phi_\ell(\sigma)$ by $\bar{\Phi}_\ell(\sigma)$ and then linearize (5.90), as it was made in the continuous case. The result turns out to be:

$$\begin{aligned}L_\ell(\tau) = \frac{L_\ell(0)(\hat{\Phi}_\ell)^2\,(1 - (\bar{\beta}_\ell)^3)}{(\Phi_\ell(\tau))^2\,(1 - (\hat{\Phi}_\ell)^3)} \frac{P(\eta_1 \Phi_\ell(\tau))}{P(\eta_1 \bar{\beta}_\ell)} \\ \times \left[1 - \int_0^\tau F(\bar{\Phi}_\ell(\sigma))\,\lambda(\sigma)\,d\sigma \right](1 + o(\gamma_N)) \qquad (5.91)\end{aligned}$$

Let us notice that, by taking N large enough, we obtain an initial distribution of radii arbitrarily uniformly close to the selfsimilar distribution in (5.2), (5.3) with probability close to one. Using continuous dependence results for the system (5.19) (see ref. 7 for some results in that direction for the classical LSW model), we would obtain that $\lambda(\tau)$ can be made smaller than γ_N for times of order one, although large if γ_N is chosen greater than the corrective term due to the large numbers law. As a matter of fact, we can assume that this approximation can be made for times large enough as to allow to approximate $\bar{\beta}_\ell$ by one for the remaining particles. With a suitable choice of γ_N such that $\gamma_N \ll 1$ but γ_N large enough, we then obtain, that $|\lambda(\tau)| \leqslant \gamma_N/10$, say, for $0 \leqslant \tau \leqslant \tau_N$, with τ_N large as $N \to \infty$, and that for $\tau \geqslant \tau_N$ we have:

$$\frac{3(\Phi_\ell(\tau))^2 L_\ell(\tau)}{P(\eta_1 \Phi_\ell(\tau))}\left[1+\int_0^\tau K(\Phi_\ell(\tau), \tau-\sigma)\, \lambda(\sigma)\, d\sigma\right] = \frac{L_\ell(0)(1+o(\gamma_N))}{\eta_1(1-a(\eta_1)^2)(1-\hat{\Phi}_\ell)} \tag{5.92}$$

where $K(\Phi, \tau-\sigma)$ is exactly as in the continuous case analysed above. Notice that (5.92) is analogous to (5.84). As in the continuous case we can introduce the change of variables $\xi_\ell = \Phi_\ell/\eta_1$. On the other hand, assuming that $\ell \gg 1$ (say larger than J_N in (5.38)), and taking into account (3.11), we can obtain an approximation $L_\ell(0)/(1-\hat{\Phi}_\ell) = (\rho_N/\alpha_1 \ell \rho_N)(1+o(\gamma_N)) = (1/\alpha_1 \ell)(1+o(\gamma_N))$. Then (5.92) becomes:

$$\frac{3a(\xi_\ell(\tau))^2 \hat{L}_\ell(\tau)}{P(\xi_\ell(\tau))}\left[1+\int_0^\tau K\left(\frac{\xi_\ell(\tau)}{\eta_1}, \tau-\sigma\right) \lambda(\sigma)\, d\sigma\right] = \frac{1}{\ell}(1+o(\gamma_N)) \tag{5.93}$$

where $\hat{L}_\ell(\tau)$ stands by the length of the intervals $[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ rewritten in the ξ-variable. In the range of times considered in this Theorem, the "differentials" $\hat{L}_\ell(\tau)$ are uniformly small, independently on γ_N. Using Taylor's theorem, as well as the differential equation for G, we can then write:

$$\delta_\ell(\ln(G(\xi))) + \int_0^\tau \lambda(\sigma)\, \delta_\ell Z(\xi, \tau-\sigma)\, d\sigma = \frac{1}{\ell}(1+o(\gamma_N)) \tag{5.94}$$

where from now on $\delta_\ell F(\xi)$ stands by $F(\xi_\ell(\tau)) - F(\xi_{\ell-1}(\tau))$. Let us denote as $\ell^* = \ell^*(\tau)$ the largest value of ℓ for which ξ_ℓ is not still zero. Adding the sequence of equations (5.94) for ℓ in a given interval (say between ℓ and ℓ^*), would then provide the formula:

$$\frac{G(\xi_\ell(\tau))}{G(\xi_{\ell^*}(\tau))}\, \frac{\exp(\int_0^\tau \lambda(\sigma)\, Z(\xi_\ell(\tau), \tau-\sigma)\, d\sigma)}{\exp(\int_0^\tau \lambda(\sigma)\, Z(\xi_{\ell^*}(\tau), \tau-\sigma)\, d\sigma)} = \left(\frac{\ell}{\ell^*}\right)^{(1+o(\gamma_N))}$$

and taking into account that $G(\xi_{\ell*}(\tau)) = 1 + o(\gamma_N)$, and $Z(\xi_{\ell*}(\theta), \tau - \sigma) = o(\gamma_N)$ with probability close to one, we deduce that:

$$S(\xi_\ell(\tau)) \equiv G(\xi_\ell(\tau)) \exp\left(\int_0^\tau \lambda(\sigma)\, Z(\xi_\ell(\tau), \tau - \sigma)\, d\sigma\right) = (1 + o(\gamma_N)) \left(\frac{\ell}{\ell*}\right)^{(1 + o(\gamma_N))} \tag{5.95}$$

Notice that $N(t)$ in (5.73) is approximately $\rho_N \ell^*(1 + o(\gamma_N))$, since the number of particles in each interval $[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ is or order ρ_N with a corrective term independent on γ_N. On the other hand, the term $o(1)$ in (5.73) is also independent on γ_N for the range of times that we are considering. Notice also that $\Phi_j = (1 + o(\gamma_N))(\xi_j/\eta_1)$, due to the probabilistic distribution of the maximum radius. We then rewrite (5.73) as:

$$v(\tau) = \frac{(1 + o(\gamma_N))}{\rho_N \ell^* \eta_1} \sum_{\{\Phi_j(t) \in \bigcup_\ell T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]\}} \xi_j(t) \tag{5.96}$$

The description of the evolution of the sets $T_\tau[\hat{\zeta}_\ell, \hat{\zeta}_{\ell-1}]$ that was given above establishes that between two values of $\xi_\ell(\tau)$ there are roughly $\rho_N(1 + o(\gamma_N))$ particles. Using this fact, as well as (5.95) and then (5.94), we can rewrite (5.96) as:

$$\begin{aligned}
v(\tau) &= \frac{(1 + o(\gamma_N))}{\ell^* \eta_1} \sum_{\ell = J_N}^{\ell^*} \xi_\ell(t) \\
&= \frac{(1 + o(\gamma_N))}{\eta_1} \sum_{\ell = J_N}^{\ell^*} \frac{\xi_\ell(t)}{\ell} \\
&\quad \times \left(G(\xi_\ell(\tau)) \exp\left(\int_0^\tau \lambda(\sigma)\, Z(\xi_\ell(\tau), \tau - \sigma)\, d\sigma\right)\right)^{1 + o(\gamma_N)} \\
&= \frac{(1 + o(\gamma_N))}{\eta_1} \cdot \sum_{\ell = J_N}^{\ell^*} \xi_\ell(t) (S(\xi_\ell(\tau)))^{1 + o(\gamma_N)} \frac{\delta_\ell(S(\xi))}{S(\xi_\ell(\tau))} \\
&= -\frac{(1 + o(\gamma_N))}{\eta_1} \cdot \sum_{\ell = J_N}^{\ell^*} \delta_\ell[\xi_\ell(t)] (S(\xi_\ell(\tau)))^{1 + o(\gamma_N)}
\end{aligned}$$

The last formula is just a Riemann approximation of the integral $\int_0^{\eta_1} (S(\xi))^{1 + o(\gamma_N)}\, d\xi$. Such approximation is independent on γ_N because it only depends on the properties of the partition of the interval $[0, \eta_1]$. Making a new linearization on $\lambda(\tau)$, we then easily obtain that:

$$\lambda(\tau) = \int_0^\tau m(\tau - \sigma)\, \lambda(\sigma)\, d\sigma + o(\gamma_N) \tag{5.97}$$

where $m(\tau)$ is as in the continuous case analysed above. Equation (5.97) can be analysed as indicated in the Appendix, Proposition A.1, to show that $|\lambda(\tau)| \leqslant o(\gamma_N)$ with probability close to one. Since all the results above were derived under the assumption that $|\lambda(\tau)| \leqslant C\gamma_N$, we then obtain the validity of this inequality for all later times. Using (5.90), the asymptotic distribution of particles then follows. ▮

6. CONCLUDING REMARKS

In this paper a model for the evolution of the distribution of radii of a set of particles, whose size changes due to the diffusion of a concentration field, has been derived. The resulting model is a correction of the classical LSW equations, that includes, in the rate of growth of the particles, the leading order of the stochastic fluctuations that are due to the probabilistic character of the distribution of particles in space. The derived model takes a simpler form if the volume fraction filled by particles is smaller that $1/N^2$. It has been recently shown (cf. ref. 8) that the long term asymptotics of the LSW system depends in a very sensitive way on the initial distribution of particles, and more precisely on the shape of this initial distribution near the maximum radius. The goal of this paper was to understand if the stochastic fluctuations mentioned above could select somehow one of the possible asymptotics among all the possibilities for the LSW model. The reason why this possibility should deserve some attention, is that the effect of the stochastic fluctuations, although small, could be in principle extremely important for radii near the maximum value. It turns out that the answer to the question above is a negative one, at least for small volume fractions. It is important to observe, that once the discrete character of the problem is taken into account the relevant problem to be considered is not the long term asymptotics, but the intermediate asymptotics for long times, but not so much as to have a dynamics dominated by fluctuations. The computations in Section 3 indicate that stochastic fluctuations should become very important as soon as the remaining number of particles is of order $\log(N)$. Before this, for times that still allow for a continuous description of the distribution of particles, the results of this paper show that the self-similar solutions that according to the classical LSW theory should be eliminated, may provide possible intermediate asymptotics for the distribution of particle sizes.

It is interesting to compare the results in this paper with those were previously obtained considering the LSW model as a limit of the Becker–Döring kinetic system. In that case there was a natural selection mechanism of the solution established in the LSW theory (cf. ref. 13). The main difference between the model considered in ref. 13 and the one studied in this paper, is that in ref. 13 nucleation effects are taken into account. In the analysis of this paper (and also refs. 5–8), particles can only disappear, but they cannot be created. The results obtained in this paper, as well as those in refs. 7, 8, 13 strongly indicate that, at least for small volume fractions, the validity of the LSW theory relies on the existence of a kinetic fluctuations in the size of the particles. Other regularization mechanisms, as for instance the collisions between growing particles that were suggested in ref. 5, occur with a too low probability for volume fractions smaller that $1/N^2$.

There are several questions that have been noticed in this paper and that deserve a more detailed analysis. Firstly, it has been assumed that the volume fraction of particles is smaller that $1/N^2$. The approximations made in Section 2.2 cease being valid for larger volume fractions. On the other hand, the smallness of this volume fraction has been repeatedly used in Section 5, in order to show that some "macroscopic differentials" of radii evolve basically without interaction with another regions. There is another interesting feature related with this fact. More precisely, there are some self-similar solutions with a singular density of particles near the maximum radius. It has been shown in Theorem 5.1 that the self-similar behaviour for the range of times considered in this paper can be proved only if the volume fraction of particles is smaller than $1/N^{(1/2)(1+3/\alpha_1)}$, where α_1 is as in (5.4). It is not clear if this critical volume fraction is the optimal one. Actually, such value appears due to the estimate (4.1) for the noise terms that are generated by stochastic fluctuations on the positions of the particles. It is very likely, however that estimates like (4.1) for only a large fraction of the particles could be obtained for larger volume fractions, and this should be enough to show results analogous to Theorem 5.1. In any case, a better understanding of the probabilistic properties of the noise terms $\lambda_j(t)$ given in (2.55) seems needed in order to clarify these point.

A problem that looks most interesting is the analysis of the dynamics driven by fluctuations, i.e., when the number of remaining particles is of order $\log(N)$ or smaller. This problem has not been addressed in this paper at all. Strictly speaking, all that has been shown in this paper is that the dynamics dominated by fluctuations begins only after that number of particles is left. It would be interesting to compute more in detail the time scale dominated by fluctuations and to describe the probabilistic properties of such regime. The estimates in Section 5 indicate that the noise terms $\lambda_j(t)$ can then be ignored, at least for initial distributions of particles that decay fast enough near the maximum radius.

Finally, throughout this paper it has been assumed that the initial distribution of particles is homogeneous in space. It would be interesting to investigate if the generation process of the distribution of particles suggests to assume the existence of correlations between the positions of the particles. Many of the computations in this paper should be made in a different way in such a case.

APPENDIX A: ANALYSIS OF AN INTEGRAL EQUATION

In this Appendix we study some properties of the integral equation (5.87). More precisely, we analyse the properties of the problem:

$$\lambda(\tau) = \int_0^{\tau} m(\tau - \sigma)\, \lambda(\sigma)\, d\sigma + g(\tau) \tag{A.1}$$

where $g(\tau)$ is a bounded function. The following result holds:

Proposition A.1. Assume that the kernel $m(\tau)$ is smooth, negative and satisfies $dm/d\tau > 0$, and $\lim_{\tau \to \infty} m(\tau) = 0$. Then, the solution of (7.1) is given by a convolution operator in the form:

$$\lambda(\tau) = \int_0^{\tau} q(\tau - \sigma)\, g(\sigma)\, d\sigma \tag{A.2}$$

where $q(\tau)$ can be bounded in the form:

$$|q(\tau)| \leqslant C e^{-\mu \tau} \tag{A.3}$$

for some $\mu > 0$.

Proof. The proof of Proposition 7.1 can be obtained by means of the standard Laplace transform. Let us denote such transform of any given function by means of capital letters. Equation (7.1) then becomes:

$$\Lambda(z) = M(z)\, \Lambda(z) + G(z) \tag{A.4}$$

Proposition 7.1 would follow as soon as we show that function $(1 - M(z))$ has not zeroes in the half-plane $\{\mathrm{Re}(z) \geqslant 0\}$. Taking into account that $M(z)$ decays like $1/|z|$ as $|z| \to \infty$, using a continuity argument, it is enough to show that function $(1-\varepsilon M(z))$ does not have a root in the line $\{\mathrm{Re}(z)=0\}$ for any $\varepsilon\in(0,1]$. By the definition of the Laplace transform, we then need to examine the roots of:

$$\Omega(\omega)=1-\varepsilon\int_0^\infty m(\tau)\,e^{i\omega\tau}\,d\tau=0,\qquad \text{for any real } \omega \tag{A.5}$$

Since $m(\tau)<0$, it follows that $\Omega(0)>0$. On the other hand, integrating by parts we readily obtain that for $\omega\neq 0$:

$$\begin{aligned}\Omega(\omega)&=1+\frac{\varepsilon}{i\omega}\int_0^\infty \frac{dm(\tau)}{d\tau}\,e^{i\omega\tau}\,d\tau+\frac{\varepsilon}{i\omega}\,m(0)\\&=1+\frac{\varepsilon}{i\omega}\int_0^\infty \frac{dm(\tau)}{d\tau}\,[e^{i\omega\tau}-1]\,d\tau\\&=1+\frac{2\varepsilon e^{i\omega\tau/2}}{\omega}\int_0^\infty \frac{dm(\tau)}{d\tau}\sin\left(\frac{\omega\tau}{2}\right)d\tau\end{aligned}$$

The imaginary part of $\Omega(\omega)$ is then given by:

$$\mathrm{Im}(\Omega(\omega))=\frac{2\varepsilon}{\omega}\int_0^\infty \frac{dm(\tau)}{d\tau}\sin^2\left(\frac{\omega\tau}{2}\right)d\tau$$

and this quantity is strictly positive for $\omega\neq 0$ under our assumptions. ∎

We show that the function $m(\tau)$ defined in Section 5 satisfies the assumptions in Proposition A.1. Taking into account the definition of $Z(\xi,\tau)$ and using (5.83), we obtain:

$$Z(\xi,\tau)=\int_0^\xi F\left(\frac{1}{\eta_1}\,G^{-1}(G(\eta)\,e^{-(\eta_1)^3\,\nu_0\tau})\right)\frac{dG(\eta)}{G(\eta)} \tag{A.6}$$

where $F(\cdot)$ is as in (5.81). We then make the change of variables $G(\eta)\,e^{-(\eta_1)^3\,\nu_0\tau}=\theta$, and use (5.88) to obtain:

$$m(\tau)=-\int_0^{\eta_1} G(\xi)\left[\int_{G(\xi)\,e^{-(\eta_1)^3\,\nu_0\tau}}^{e^{-(\eta_1)^3\,\nu_0\tau}} F\left(\frac{1}{\eta_1}\,G^{-1}(\theta)\right)\frac{d\theta}{\theta}\right]d\xi \tag{A.7}$$

that immediately implies that $m(\tau)$ is negative. It is easily seen that $m(\tau)$ exponentially approaches to zero. Finally, differentiating (A.7), it turns out that:

$$\frac{dm(\tau)}{d\tau} = (\eta_1)^3 \nu_0 \int_0^{\eta_1} G(\xi) \left[F\left(\frac{1}{\eta_1} G^{-1}(e^{-(\eta_1)^3 \nu_0 \tau}) \right) - F\left(\frac{1}{\eta_1} G^{-1}(G(\xi)\, e^{-(\eta_1)^3 \nu_0 \tau}) \right) \right] \tag{A.8}$$

Simple computations show then that:

$$F(x) = \frac{(1 + x - 2x^2)}{[(\nu_0 - 1)\, x^2 + (\nu_0 - 1)\, x + \nu_0]\, x^2} \tag{A.9}$$

Differentiating (A.9) it readily follows that $F(x)$ is decreasing. Since G^{-1} is also decreasing, it follows that $F((1/\eta_1)\, G^{-1}(\cdot))$ is increasing. Taking into account that $G(\xi) \leqslant 1$ it then follows from (A.8) that $dm(\tau)/d\tau > 0$, whence the desired monotonicity follows.

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